Integral Formulation for Numerical Solutions

**OBJECTIVE:** Use integral methods to obtain an approximate solution to a physical problem.

**EXAMPLE:** Find a solution for a beam deflection.

\[ H(x) = m_0 \]

\[ M(x) = \frac{m_0}{EI} \]

\[ EI \frac{d^2y}{dx^2} - M(x) = 0 \]

- **EI** = beam stiffness
- **M(x)** = bending moment

Differential Equations

\[ EI \frac{d^2y}{dx^2} - M(x) = 0 \]
Approximate Beam deflection:

\[ y(x) = \sin \left( \frac{\pi x}{H} \right) \]

Exact Solution:

\[ \frac{d^2 y}{dx^2} = \frac{M(x)}{EI} \]

Integrating twice

\[ \frac{dy}{dx} = \frac{Mx}{EI} + c \]

\[ y = \frac{Mx^2}{2EI} + c_1 x + c_2 \]

Using boundary conditions:

\[ y(0) = y(H) = 0 \]

\[ y(0) = 0, x = 0 \implies c_2 = 0 \]

\[ 0 = \frac{M H^2}{2EI} + c_1 H \]

\[ c_1 = -\frac{MH^2}{2EIH} = \frac{MH}{2EI} \]

\[ y = \frac{Mx^2}{2EI} - \frac{MHx}{2EI} \]

\[ y = \frac{MHx}{2EI} (x - H) \quad \text{Exact Solution.} \]
VARIATIONAL METHOD

Given a differential equation such as:

\[ D \frac{d^2 y}{dx^2} - Q = 0 \]

The calculus of variations will generate a solution if the numerical value of the integral \( \Pi \) is a minimum.

In other words, if a particular \( y = g(x) \) generates a minimum for \( \Pi \); it is the solution of the differential equation.
So, for the beam:

\[ \Pi = \int_0^H \left[ \frac{EI}{2} \left( \frac{dy}{dx} \right)^2 - M_0 y \right] dx \]

Consider the approximate function, \( y(x) = A \sin \frac{\pi x}{H} \), then evaluate \( \Pi \).

**Steps:**

1. Write \( \Pi \) as a function of \( A \)
2. Minimize with respect to \( A \)

\[ y(x) = A \sin \frac{\pi x}{H} \]

\[ \frac{dy}{dx} = A \frac{\pi}{H} \cos \frac{\pi x}{H} \]
VARIATIONAL METHOD
continued

\[ \Pi = \int_0^H \left[ \frac{EI}{2} \left( \frac{A\pi}{H} \cos \frac{\pi x}{H} \right)^2 - M_0 A \sin \frac{\pi x}{H} \right] \, dx \]

\[ \Pi = \left( \frac{EI\pi^2}{4H} \right) A^2 + \left( \frac{2M_0 H}{\pi} \right) A \]
Minimizing $\Pi$:

$$\frac{\partial \Pi}{\partial A} = 2 \left( \frac{EI\pi^2}{4H} \right) A + \frac{2M_0 H}{\pi} = 0$$

$$A = -\frac{4M_0 H^2}{\pi^3 EI}$$

$$y(x) = -\frac{4M_0 H^2}{\pi^3 EI} \sin \frac{\pi x}{H}$$
Development of a Continuous Solution in a 1 dimensional region

\[ \phi^{(1)} = N_i^{(1)} \Phi_i + N_j^{(1)} \Phi_j = N_1^{(1)} \Phi_1 + N_2^{(1)} \Phi_2 \]
\[ \phi^{(2)} = N_i^{(2)} \Phi_i + N_j^{(2)} \Phi_j = N_2^{(2)} \Phi_2 + N_3^{(2)} \Phi_3 \]
\[ \phi^{(3)} = N_i^{(3)} \Phi_i + N_j^{(3)} \Phi_j = N_3^{(3)} \Phi_3 + N_4^{(3)} \Phi_4 \]
\[ \phi^{(4)} = N_i^{(4)} \Phi_i + N_j^{(4)} \Phi_j = N_4^{(4)} \Phi_4 + N_5^{(4)} \Phi_5 \]

Apply shape functions concept to each element
\[
\phi^{(e)} = N_i^{(e)} \Phi_i + N_j^{(e)} \Phi_j
\]

General Form

With Shape functions:

\[
N_i^{(e)} = \frac{X_j - x}{X_j - X_i} \quad \& \quad N_j^{(e)} = \frac{x - X_i}{X_j - X_i}
\]
Development of a Continuous Solution in a 1 dimensional region (continued)

Grid Summary:

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<tr>
<th>element</th>
<th>nodes</th>
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Note:
\[ N_2^{(1)} \neq N_2^{(2)} \]