USE OF GALERKIN’S METHOD FOR THE SOLUTION OF ONE DIMENSIONAL DIFFERENTIAL EQUATIONS

**Problem:** Solve equations of the form:

\[ D \frac{d^2 y}{dx^2} - Q = 0 \]

Solid mechanics problems
Solution of Differential Equations
- Beam Deflection
- Heat flow through a wall

With boundary conditions:

\[ \phi(0) = \phi_0 \quad \& \quad \phi(H) = \phi_H \]
**Procedure:** Evaluate Residual Integral.

\[ \int_{0}^{H} W_i(x) R(x) \, dx = 0 \]

- Obtain a nodal equation for each node
- Apply integral evaluation to each node
- Generate a system of linear equations
- Find beam deflection
USE OF GALERKIN’S METHOD FOR THE SOLUTION
OF ONE DIMENSIONAL DIFFERENTIAL EQUATIONS

DEFINE WEIGHTING FUNCTIONS:

- Evaluate the Weighted Residual Integral

\[- \int_{0}^{H} W(x) \left( D \frac{d^2 \phi}{dx^2} + Q \right) dx = 0\]

\[R(x) \quad \text{because } \phi \text{ is not an exact solution}\]
- Use a new weighting function for each node where $\phi$ is unknown.
- Construct weighting functions using the shape functions $N_i$ & $N_j$.
- Weighting functions of the $s^{th}$ node = $W_s$, or $W_s = \text{shape functions associated with the } s^{th} \text{ node}.

$$W_3(x) = \begin{cases} N_3^{(2)} & x_2 \leq x \leq x_3 \\ N_3^{(3)} & x_3 \leq x \leq x_4 \end{cases}$$

$$W_s(x) = \begin{cases} N_s^{(e)} & x_r \leq x \leq x_s \\ N_s^{(e+1)} & x_s \leq x \leq x_t \end{cases}$$
EVALUATION OF THE RESIDUAL INTEGRAL:

We need to evaluate
\[-\int_0^H W(x) \left( D \frac{d^2 \phi}{dx^2} + Q \right) dx = 0\]

\[R_s = R_s^{(e)} + R_s^{(e+1)}\]

Residual due to element \((e)\)  \hspace{1cm} Residual due to element \((e+1)\)

\[-\int_{x_r}^{x_s} N_s \left( D \frac{d^2 \phi}{dx^2} + Q \right) dx^{(e)} - \int_{x_s}^{x_t} N_s \left( D \frac{d^2 \phi}{dx^2} + Q \right) dx^{(e+1)} = 0\]

Since \(\phi\) is it is not continuous

\[\therefore \int \frac{d^2 \phi}{dx^2} dx \text{ not defined}\]
But it can be changed to a new term such as:

\[
\frac{d}{dx} \left( N_s \frac{d\phi}{dx} \right) = N_s \frac{d^2\phi}{dx^2} + \frac{dN_s}{dx} \frac{d\phi}{dx}
\]

Then:

\[
N_s \frac{d^2\phi}{dx^2} = \frac{d}{dx} \left( N_s \frac{d\phi}{dx} \right) - \frac{dN_s}{dx} \frac{d\phi}{dx}
\]

Now substitute into the integral:
EVALUATION OF THE RESIDUAL INTEGRAL:

(continued)

\[- \int_{x_r}^{x_t} \left( N_s D \frac{d^2 \phi}{dx^2} \right)^{(e)} \, dx = - \left( D N_s \frac{d \phi}{dx} \right)^{(e)} \bigg|_{x_r}^{x_t} + \int_{x_r}^{x_t} \left( D \frac{d N_s}{dx} \frac{d \phi}{dx} \right)^{(e)} \, dx \]

\[- \int_{x_s}^{x_t} \left( N_s D \frac{d^2 \phi}{dx^2} \right)^{(e+1)} \, dx = - \left( D N_s \frac{d \phi}{dx} \right)^{(e+1)} \bigg|_{x_s}^{x_t} + \int_{x_s}^{x_t} \left( D \frac{d N_s}{dx} \frac{d \phi}{dx} \right)^{(e+1)} \, dx \]

Now we need to evaluate the complete residual.
Complete residual equation: (Interior node)

\[ R_s = R_s^{(e)} + R_s^{(e+1)} = - \int_0^H W(x) \left( D \frac{d^2 \phi}{dx^2} + Q \right) \, dx \]
\[ R_s = R_s^{(e)} + R_s^{(e+1)} = - \int_0^H W(x) \left( D \frac{d^2 \phi}{dx^2} + Q \right) dx \]

\[ = - \left( D \cdot 1 \frac{d\phi}{dx} \right)^{(e)} \bigg|_{x=x_s} + \int_{x_s}^{x_r} \left( D \frac{dN_s}{dx} \frac{d\phi}{dx} - N_s Q \right)^{(e)} dx \]

Remember
\[ N_i = 1 \quad N_j = 0 \]
\[ N_s = 1 \quad N_r = 0 \]

Evaluation of these establishes an interelement requirement.
Difference must be zero so that \( R_s = 0 \).

\[ + \left( D \cdot 1 \frac{d\phi}{dx} \right)^{(e+1)} \bigg|_{x=x_s} + \int_{x_s}^{x_t} \left( D \frac{dN_s}{dx} \frac{d\phi}{dx} - N_s Q \right)^{(e+1)} dx = 0 \]

and
\[ N_s = 1 \quad N_t = 0 \]
EVALUATION OF THE INTEGRAL:
(continued)

Remember

\[ \phi^{(e)} = N_r \Phi_r + N_s \Phi_s \]

Shape Functions

\[ \phi^{(e)} = \left( \frac{X_s - x}{L} \right) \Phi_r + \left( \frac{x - X_r}{L} \right) \Phi_s \]

Thus:

\[ N_s^{(e)} = \frac{x - X_r}{L} \Rightarrow \frac{dN_s^{(e)}}{dx} = \frac{1}{L} \]
\[
\dot{\phi}^{(e)} = \frac{1}{L} (-\Phi_r + \Phi_s)
\]

This will give the rate of change of \( \Phi \) within the element.

So, we want \( R_s^{(e)} \) & \( R_s^{(e+1)} \)

\[R_s^{(e)} = -\left( D \cdot \frac{d\phi}{dx} \right)^{(e)} \bigg|_{x=x_s} + \int_{x_r}^{x_s} \left( D \frac{dN_s}{dx} \frac{d\phi}{dx} - N_s Q \right)^{(e)} \ dx\]

\[= -\left( D \cdot \frac{d\phi}{dx} \right)^{(e)} \bigg|_{x=x_s} + \int_{x_r}^{x_s} \left( D \frac{dN_s}{dx} \frac{d\phi}{dx} \right)^{(e)} \ dx - \int_{x_r}^{x_s} N_s Q \ dx\]
EVALUATION OF THE INTEGRAL:

(continued)

\[ R_s^{(e)} = -\left( D \cdot 1 \cdot \frac{d\phi}{dx} \right)^{(e)} \bigg|_{x=x_s} + \int_{x_r}^{x_s} \left( D \frac{dN_s}{dx} \frac{d\phi}{dx} \right)^{(e)} dx - \int_{x_r}^{x_s} N_s Q \, dx \]

\[ + \int_{x_r}^{x_s} \left( D \left( \frac{1}{L} \right) \left( \frac{1}{L} \right) (-\Phi_r + \Phi_s) \right) dx - \int_{x_r}^{x_s} Q \left( \frac{x - X_r}{L} \right) dx \]

\[ + D \left( \frac{1}{L} \right) (-\Phi_r + \Phi_s) - \frac{Q}{L} \left( \frac{x^2}{2} - X_r \cdot x \right) \bigg|_{x=X_s} \]

\[ R_s^{(e)} = -\left( D \cdot 1 \cdot \frac{d\phi}{dx} \right)^{(e)} \bigg|_{x=x_s} + \frac{D}{L} (-\Phi_r + \Phi_s) - \frac{QL}{2} \]
EVALUATION OF THE INTEGRAL:

(continued)

Similarly, we obtain \( R_s^{(e+1)} \)

\[
\phi^{(e+1)} = N_s \Phi_s + N_t \Phi_t
\]

\[
\phi^{(e+1)} = \left( \frac{X_t - x}{L} \right) \Phi_s + \left( \frac{x - X_t}{L} \right) \Phi_t
\]

\[
N_s^{(e+1)} = \frac{X_t - x}{L} \Rightarrow \frac{dN_s^{(e+1)}}{dx} = -\frac{1}{L}
\]

\[
\frac{d\phi^{(e+1)}}{dx} = \frac{1}{L} (-\Phi_s + \Phi_t)
\]
\[ R_s^{(e+1)} = -\left( D \cdot \frac{d\phi}{dx} \right)^{(e+1)} \bigg|_{x=x_s} + \int_{x_s}^{x_t} \left( D \frac{dN_s}{dx} \frac{d\phi}{dx} - N_s Q \right)^{(e+1)} \, dx \]

\[ + \int_{x_s}^{x_t} \left( D \left( -\frac{1}{L} \right) \left( \frac{1}{L} \right) (-\Phi_s + \Phi_t) \right) dx - \int_{x_s}^{x_t} Q \left( \frac{X_t - x}{L} \right) dx \]

\[ + \left( \frac{D}{L} \right)(\Phi_s - \Phi_t) - \frac{QL}{2} \]
EVALUATION OF THE INTEGRAL:
(continued)

Finally the residual equation for node “s”:

\[ R_s = \left( D \frac{d\phi}{dx} \right)^{(e+1)} \bigg|_{x=x_s} - \left( D \frac{d\phi}{dx} \right)^{(e)} \bigg|_{x=x_s} \]

Interelement requirement

\[-\frac{D}{L}^{(e)} \Phi_r + \left[ \frac{D}{L}^{(e)} + \frac{D}{L}^{(e+1)} \right] \Phi_s - \frac{D}{L}^{(e+1)} \Phi_t \]

\[-\left( \frac{QL}{2} \right)^{(e)} - \left( \frac{QL}{2} \right)^{(e+1)} = 0 \]
Observation about the interelement requirement.
- Can be used to evaluate quality of grid.
- Indicate where grid should be refined.
- If \( D^{(e)} = D^{(e+1)} \) \( \Rightarrow \left( \frac{d\phi}{dx} \right)^{(e+1)} \bigg|_{x=X_s} - \left( \frac{d\phi}{dx} \right)^{(e)} \bigg|_{x=X_s} \)
- It can be viewed as an error term.
- Not incorporated into system of equations.
- Remind us that solution is approximate.

So finally the nodal residual equation is:

\[
R_s = -\left(\frac{D}{L}\right)^{s-1} \Phi_{s-1} + \left[ \frac{D^{(s-1)}}{L} + \frac{D^{(s)}}{L} \right] \Phi_s - \frac{D^{(s)}}{L} \Phi_{s+1} - \left(\frac{QL}{2}\right)^{(s-1)} - \left(\frac{QL}{2}\right)^{(s)} = 0
\]

Nodal Residual equation for node “s”