First and second order systems are common in real world engineered and physical systems. The transfer function of a second order system is given by

\[ \frac{b}{s^2 + as + b} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]  

(1)

The transient behavior of second order systems can be described by two variables

- Natural frequency \( \omega_n \): frequency of oscillations without damping. In series RLC circuit, the natural frequency is obtained by shorting the resistor.
- Damping ratio \( \zeta \): The damping ratio compares the exponential decay to the natural frequency of oscillations. Damping reduces the amplitude of oscillations. It results from forces that oppose motion and dissipate energy.

Depending on the location of the poles, we have four different cases:

- undamped (poles are pure imaginary) \( \zeta = 0 \)
- underdamped (complex conjugate) \( 0 < \zeta < 1 \)
- critically damped (repeated poles) \( \zeta = 1 \)
- overdamped (poles are real) \( \zeta > 1 \)

**Underdamped Second Order System**

We defined two parameters for second order systems: \( \zeta, \omega_n \). Based on these variables, it is possible to define other parameters:

- Rise time: time to go from 10% to 90% of the final value.
- Peak time: time required to reach the first or maximum peak
- Percent overshoot: the amount that the time response overshoots the steady state at the peak time
- Settling time: the time required for the signal to reach and stay within 2 percent of its final value

The equations are

\[ T_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} \]  

(2)

\[ \%OS = \frac{c_{\text{max}} - c_{\text{final}}}{c_{\text{final}}} = e^\frac{-\pi \zeta}{\sqrt{1 - \zeta^2}} \]  

(3)

\[ T_s = \frac{4}{\zeta \omega_n} \]  

(4)

There is no exact analytic expression for \( T_r \).

**A. Example**

Let

\[ G(s) = \frac{9}{s^2 + 3s + 9} \]  

(5)

Find the percent overshoot, the settling time and the peak time.

**B. Solution**

Simply use the formulas. We have \( \omega_n = 3, \zeta = 0.5 \)

**C. Example**

Given the location of the poles in figure 4, find the percent overshoot, the settling time and the peak time.

**D. Solution**

The poles are given by

\[ s_{1,2} = -\zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1} \]  

First, we find \( \zeta \) and \( \omega_n \) by solving

\[ -3 \pm j7 = -\zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1} \]  

(7)

and then use the equations to solve.

**Systems with Additional Poles**

The formulas we have written for the percent overshoot, the settling time and the peak time are derived for a particular case where the second order system has no zeros (and exactly two poles). Sometimes, systems with more poles can be approximated by first or second order systems (because they behave like first or second order systems). These systems have dominant poles (poles that are the closest to the \( j\omega \) axis). Consider the following example
E. Example

We have the following transfer function:

\[ G(s) = \frac{3000}{(s + 1)(s + 100)(s + 1000)} \]  
(8)

The output to a step input is

\[ C(s) = \frac{3000}{s(s + 1)(s + 10)(s + 100)} \]  
(9)

and the partial fraction expansion gives us:

\[ C(s) = \frac{a}{s} + \frac{b}{s + 1} + \frac{c}{s + 10} + \frac{d}{s + 100} \]  
(10)

and the step response is

\[ c(t) = a + be^{-t} + ce^{-10t} + de^{-100t} \]  
(11)

The first term is the forced response, it will stay there for ever (as long as the input remains equal to 1). The other terms correspond to the natural response. The last terms \( ce^{-10t} \) and \( de^{-100t} \) will die out much faster, and the dominant pole is \(-1\).

Once the dominant pole has been identified, we can simplify the transfer function by

- Keeping the dominant poles only
- Making sure the DC gain is unchanged

F. Example

Find a simplified model for the transfer function:

\[ G(s) = \frac{3000}{(s + 1)(s + 10)(s + 100)} \]  
(12)

G. Solution

The dominant pole is \(-1\) from the previous example. Thus the first order approximation is

\[ G_{\text{approximation}}(s) = \frac{M}{s + 1} \]  
(13)

where \( M \) is a constant. We need to determine \( M \) so that the DC gain is the same. The DC gain can be obtained by setting \( s = 0 \), therefore:

\[ DC_{\text{gain}} = \frac{3000}{(0 + 1)(0 + 10)(0 + 100)} = 3 \]  
(14)

and for the first order system:

\[ DC_{\text{gain}} = \frac{M}{0 + 1} = M \]  
(15)

Thus, the first order system approximation is

\[ G_{\text{approximation}}(s) = \frac{3}{s + 1} \]  
(16)

H. Example

Assume a system has three poles: two complex conjugate \( \sigma \pm j\omega \) and one real pole \( \alpha \). The step response is given by:

\[ c(t) = A + Be^{-\alpha t} \cos(\omega t + \phi) + e^{-\alpha t} \]  
(17)

We consider four cases as shown in figure 2:

- Case I: \( \alpha \) is comparable to \( \sigma \): Third order system approximation by lower order systems is not valid.
- Case II: \( \alpha >> \sigma \): System can be approximated by a second order system
- Case III: \( \alpha = \infty \): This is a second order system.
- Case IV: \( \alpha << \sigma \): System can be approximated by a first order system

I. Example

In this case we consider the following transfer function for three different values of \( a \).

\[ G(s) = \frac{1}{(s + 1 + j2)(s + 1 - j2)(s + a)} \]  
(18)

- \( a = 0.1 \): In this case \( a \) is the dominant pole. The system behaves like a first order system.
- \( a = 1 \): There is no dominant pole.
- \( a = 10000 \): In this case \( 1 \pm 2j \) are the dominant poles. The system behaves like a second order system.

The time response for different values of \( a \) is shown in figures 3 and 4.

LAPLACE TRANSFORM SOLUTION OF STATE SPACE

The goal is to find the time solution of a system i.e., state and output \((x(t), y(t))\) by using the Laplace transform and its properties. We already know that

\[ \dot{x} = Ax + Bu \]  
(19)
\[ y = Cx + Du \]  
(20)

We can write

\[ sX(s) - X(0) = AX(s) + BU(s) \]  
(21)
\[ (sI - A)X(s) = X(0) + BU(s) \]  
(22)
\[ X(s) = (sI - A)^{-1}(X(0) + BU(s)) \]  
(23)
The inverse of \((sI - A)\) is given by

\[
(sI - A)^{-1} = \frac{1}{\det} \begin{bmatrix} s + 5 & 2 \\ -3 & s \end{bmatrix}
\]

(28)

\[
(sI - A)^{-1} = \begin{bmatrix} \frac{s + 5}{s^2 + 5s + 6} & \frac{s + 5}{s^2 + 5s + 6} \\ \frac{s^2 + 5s + 6}{s^3 + 7s^2 + 2} & \frac{s^2 + 5s + 6}{s^3 + 7s^2 + 2} \end{bmatrix}
\]

(29)

and

\[
X(s) = (sI - A)^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{0}{s}
\]

(30)

\[
X(s) = \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} \frac{s^2 + 7s + 2}{s(s^2 + 5s + 6)} \\ \frac{s^2 + 7s + 2}{s(s^2 + 5s + 6)} \end{bmatrix}
\]

(31)

Now we use partial fraction expansion:

\[
X_1(s) = \frac{-3.33}{s + 3} + \frac{4}{s + 2} + \frac{0.33}{s}
\]

(32)

and

\[
X_2(s) = \frac{5}{s + 3} - \frac{4}{s + 2}
\]

(33)

and therefore:

\[
x_1(t) = -3.33e^{-3t} + 4e^{-2t} + 0.33
\]

(34)

\[
x_2(t) = 5e^{-3t} - 4e^{-2t}
\]

(35)

and for the output

\[
y(t) = 0x_1(t) + 2x_2(t) = 10e^{-3t} - 8e^{-2t}
\]

(36)

TIME DOMAIN SOLUTION OF STATE SPACE

It can be proven that the solution for \(x(t)\) has the following general form

\[
x(t) = e^{At}x(0) + \int_{0}^{t} e^{A(t-\tau)} Bu(\tau)d\tau
\]

(37)

The first term in this equation depends on the initial condition and is called the zero input response. The second term is a
convolution integral, it depends on the input and it is called zero state response. It is important to note that $e^{At}$ is the exponential of a matrix and is also a matrix ($expm$ in Matlab).

**NUMERICAL SOLUTION**

Matlab has various functions that can be used to solve state space models. First you need to declare the system as a state space model. This can be done using command ss as follows:

$$\texttt{mysystem} = \texttt{ss}(A, B, C, D)$$  \hspace{1cm} (38)

After declaration you can use one of the following functions:

- **step(mysystem)**: This function assumes zero initial conditions
- **initial(mysystem, x0)**: This function assumes zero input
- **lsim**: Can be used to solve for the general case. The syntax for this function is

$$\texttt{lsim(mysystem, u, t, x0)}$$  \hspace{1cm} (39)

For our previous example, we have

$$x0 = [1; 1]; \hspace{1cm} \%\text{Initial condition}$$  \hspace{1cm} (40)

$$t = 0 : 0.01 : 5; \hspace{1cm} \%\text{5 is the final time}$$  \hspace{1cm} (41)

$$u = \texttt{ones(size(t))} \hspace{1cm} \%\text{Unit step with the same size as} \ t$$  \hspace{1cm} (42)

$$\texttt{lsim(mysystem, u, t, x0)}$$  \hspace{1cm} (43)

which allows us to get the solutions shown in figures 5 and 6.