Digital Control System

Summary #2

The z-transform plays an important role in digital control and discrete signal processing. The z-transform is defined as

$$F(z) = \sum_{k=0}^{\infty} f(k)z^{-k}$$  \hspace{1cm} (1)

A. Example

Consider the following sequence:

$$f(k) = (0.5)^k, \quad k \geq 0$$  \hspace{1cm} (2)

The corresponding z-transform is given by

$$F(z) = \sum_{k=0}^{\infty} (0.5)^k z^{-k} = \frac{z}{z - 0.5}$$  \hspace{1cm} (3)

In general, if

$$f(k) = a^k, \quad k \geq 0$$  \hspace{1cm} (4)

then

$$F(z) = \frac{z}{z - a}$$  \hspace{1cm} (5)

B. The final value theorem

The final value theorem states that:

$$\lim_{z \to 1} (z-1)F(z) = f(\infty)$$  \hspace{1cm} (6)

The final value theorem allows to obtain the final value of a sequence without solving for the time response.

C. Example: transfer function from difference equation

Find the transfer function for the following system

$$y(k + 1) - y(k) = u(k + 1)$$  \hspace{1cm} (7)

Using the time advance property of the z-transform, it is possible to write

$$zY(z) - Y(z) = zU(z)$$  \hspace{1cm} (8)

and therefore, we obtain for the transfer function

$$Y(z) = \frac{z}{z - 1}$$  \hspace{1cm} (9)

which usually gives the relationship between the input and the output.

D. Time response of a discrete time system

The general form for discrete time dynamical system is given by

$$x(k + 1) = ax(k) + bu(k)$$  \hspace{1cm} (15)

where $x$ and $u$ are the state and input variables, respectively. The solution for system (15) can be obtained using the following formula

$$x(k) = a^k x(0) + \sum_{n=0}^{k-1} a^{k-n-1}bu(n)$$  \hspace{1cm} (16)

The time response has two components:

- Natural response: characterized by the initial condition, i.e., the first term in the right hand side in equation (16).
- Forced response: characterized by the input, i.e., the second term in the right hand side in equation (16).

For a linear time invariant system such as the one in figure 1, the relationship between the input and the output is given by:

$$y(k) = h(k) * u(k) = \sum_{i=0}^{\infty} h(k - i)u(i)$$  \hspace{1cm} (17)

This operation is called convolution and $h$ is called the impulse response of the system (response to an impulse). The convolution becomes a simple multiplication in the z-domain. That is:

$$Y(z) = H(z)U(z)$$  \hspace{1cm} (18)

and $y(k)$ can be obtained from $Y(z)$ using the inverse z-transform.
E. Example

Find the impulse response of the system

\[ y(k + 1) - 0.5y(k) = u(k) \]
\[ y(0) = 0 \]

- The unit impulse is defined as
  \[ u(k) = \delta(k) = \begin{cases} 
  1 & \text{for } k = 0 \\
  0 & \text{for } k \neq 0 
\end{cases} \]

We get

\[ y(1) = 0.5 \times 0 + 1 \]
\[ y(2) = 0.5 \times 0 + 0.5 \times 1 + 0 \]
\[ y(3) = 0.5^2 \]
\[ y(4) = 0.5^3 \]
\[ y(5) = 0.5^4 \]

which leads us to:

\[ y(k) = 0.5^{k-1} \]

- Using the z-transform, we get
  \[ zY(z) - 0.5Y(z) = U(z) \]
from which we get

\[ Y(z) = \frac{U(z)}{z - 0.5} \]

Since the input is a unit impulse

\[ Y(z) = \frac{1}{z - 0.5} \]

We already know that

\[ a^k \Leftrightarrow \frac{z}{z-a} \]

The time delay property implies that:

\[ Z\{ f(k-n) \} = z^{-n} F(z) \]

Using the previous equations including the time delay property with \( n = 1 \), we get

\[ y(k) = 0.5^{k-1} \]

which confirms the previous result.

F. Example: Final value of a sequence

Consider the system

\[ y(k + 1) - 0.75y(k) = u(k) \]
\[ y(0) = 0 \]

where \( u(k) \) is a unit step. Find \( y(\infty) \) using the time response and then using the final value theorem.

- Using the time response: partial fraction expansion allows us to write
  \[ Y(z) = \frac{1}{(z - 0.75)(z - 1)} = \frac{4}{(z - 1)} + \frac{-4}{(z - 0.75)} \]
Therefore

\[ y(k) = 4(1)^k - 4(0.75)^k \]
as \( k \) approaches infinity, \((0.75)^k \to 0\), and thus:

\[ y(\infty) = 4 \]

- Using the final value theorem
  \[ Y(z) = \frac{U(z)}{z - 0.75} \]
Since the input is a unit step:

\[ Y(z) = \frac{z}{(z - 0.75)(z - 1)} \]

Now

\[ y(\infty) = \lim_{z \to 1} \frac{z}{(z - 0.75)(z - 1)} = 4 \]
The unit step response for system (35) is shown in figure 2. Clearly, from the figure, the final value is 4.

G. Forward and backward Euler approximation

- Forward Euler approximation: We approximate the derivative at sample \( k \) by looking forward and comparing
between current (at time \(k\)) and next sample (at time \(k + 1\)). This gives us
\[
\dot{y}(k) = \frac{y(k+1) - y(k)}{T} \quad (43)
\]

- Backward Euler approximation: We approximate the derivative at sample \(k\) by looking backward and comparing between current \((k)\) and past sample \((k-1)\). This implies
\[
\dot{y}(k) = \frac{y(k) - y(k-1)}{T} \quad (44)
\]
Equation (44) is then written as
\[
\dot{y}(k+1) = \frac{2}{T} (y(k+1) - y(k)) \quad (45)
\]

- Taking the average: By averaging the forward and backward approximations we obtain:
\[
\dot{y}(k+1) + \dot{y}(k) = \frac{2}{T} (y(k+1) - y(k)) \quad (46)
\]

By introducing the \(s\) and \(z\) variables in equations (43, 44, 46), it is possible to write:

- From equation (43):
\[
sY = \frac{zY - Y}{T} \quad (47)
\]
from which, we obtain:
\[
s = \frac{z - 1}{T} \quad (48)
\]

- From equation (44):
\[
szY = \frac{zY - Y}{T} \quad (49)
\]
from which, we obtain:
\[
s = \frac{z - 1}{Tz} \quad (50)
\]

- From equation (46):
\[
szY + sY = \frac{2}{T} (zY - Y) \quad (51)
\]
from which, we obtain:
\[
s = \frac{2z - 1}{Tz + 1} \quad (52)
\]
This transformation is called bilinear or Tustin transformation. Equations (48, 50, 52) allow to obtain a digital approximation of continuous time transfer functions.

### H. Tustin transformation: \(s\) to \(z\)-plane

The transformation between the \(s\)-plane and the \(z\)-plane is given by
\[
z = e^{sT} \quad (53)
\]
The Tustin transformation is a linear approximation of this relationship. Solving for \(s\) as a function of \(z\), we get
\[
s = \frac{\ln z}{T} \quad (54)
\]

The natural logarithm can be expanded into an infinite series as follows
\[
\ln z = 2(x + 1/3x^3 + 1/5x^5 + \ldots) \quad (55)
\]
where
\[
x = \frac{1 - z^{-1}}{1 + z^{-1}} \quad (56)
\]
The Tustin transformation is obtained by keeping the first term only:
\[
s = \frac{2z - 1}{Tz + 1} \quad (57)
\]
or
\[
z = \frac{1 + sz}{1 - sz} \quad (58)
\]
The transformation maps the left half plane in the \(s\)-domain to the unit disk in the \(z\)-plane as shown in figure 3- top. Tustin method is also called the bilinear transform. It is possible to use Matlab function
\[
\text{sysd} = \text{c2d(sys,Ts,\text{method})} \quad (59)
\]
to find the discrete time system where \(Ts\) is the sampling time, for Tustin approximation \text{method} = ‘tustin’.

### I. Modeling of Digital Control Systems

The block diagram of a digital control system is shown in figure 5, where

- DAC converts numbers calculated by the micro controller into analog signals
- The analog subsystem includes the plant, amplifiers, actuators, etc.
- The output of the analog system is measured and converted into a number fed back to the microcontroller.

#### A. ADC model

We assume that there is no delay and the sampling is uniform, this implies a fixed sampling rate. These assumptions
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Figure 4. Relationship between continuous (differential equations), discrete (difference equations) and frequency domain representations.

Figure 5. Top: block diagram for a digital control system, middle: an ideal sampler, and bottom zero order hold.

are reasonable and accepted for most engineering applications. The ideal sampler of period $T$ is just a switch. Ideal sampler implies that the switch closure time is much smaller than the sampling period. Ideal sampling is also called impulse sampling because it can be modeled as an impulse train as follows:

$$
\delta_T(t) = \sum_{k=0}^{\infty} \delta(t - kT) \quad (60)
$$

where $\delta(t - kT)$ is a delayed impulse. The sampled signal becomes

$$
f(kT) = f(t)\delta_T(t) = \sum_{k=0}^{\infty} f(t)\delta(t - kT) \quad (61)
$$

where $f(kT)$ represents $f(t)$ at sampling time $kT$.

B. DAC model

Continuous signal reconstruction is achieved by the DAC. We want to find an input-output relationship for the DAC. The zero-order hold (ZOH) is the mathematical model that allows modeling the conventional digital-to-analog converter (DAC). The ZOH reconstructs the analog signal by holding each sample value for one sampling period:

$$
\{u(k)\} \Rightarrow u(t) = u(k) \quad \text{for} \quad kT \leq t \leq (k+1)T \quad (62)
$$

Zero order hold is the most widely used technique, but first order hold and second order hold are also used.

- First order hold uses a straight line as shown in Figure 6.
- Second order hold uses a parabola as shown in Figure 6.

The transfer function of a zero order hold can be obtained noting that a rectangular pulse can be represented by a positive step followed by a negative step (Figure 6–bottom). We already know that

$$
\mathcal{L}\{u(t)\} = \frac{1}{s} \quad (63)
$$

where $u(t)$ is the unit step. Using Laplace transform properties, we can write

$$
\mathcal{L}\{u(t-T)\} = \frac{e^{-sT}}{s} \quad (64)
$$

Therefore

$$
G_{zoh}(s) = \frac{1 - e^{-sT}}{s} \quad (65)
$$

C. DAC, analog subsystem and ADC

Cascading the DAC, analog system and ADC appears frequently in digital control systems. The goal here is to derive discrete time transfer function of the entire system.
D. Example

Consider the circuit in figure 7, the goal is to find the digital transfer function of the system. It is possible to write

$$G_{za}(s) = G(s)G_{zoh}(s) = \left(1 - e^{-sT}\right) \frac{G(s)}{s}$$

and

$$g_{za}(t) = g(t) * g_{zoh}(t)$$

from which it is possible to write

$$G(z)_{za} = \left(1 - z^{-1}\right) \mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$


E. Example

Find $G_{za}$ knowing the analog system is given by the circuit of figure 7.

We have

$$G(s) = \frac{\tau s}{1 + \tau s}$$

Therefore,

$$G_{za} = \left(1 - z^{-1}\right) \mathcal{Z}\left\{\frac{1}{s + 1/\tau}\right\}$$

with $\tau = L/R$. From the table it is possible to write

$$G_{za} = \frac{z - 1}{z} \frac{z}{z - e^{-T/\tau}}$$

$$G_{za} = \frac{z - 1}{z - e^{T/\tau}}$$

We can also use the Tustin transform. In this case, we have

$$G_{za} = \frac{\tau^{2(z-1)}}{1 + \frac{2\tau(z-1)}{T(z+1)}}$$

$$G_{za} = \frac{2\tau z - 2\tau}{z(T + 2\tau) + T - 2\tau}$$

II. CLOSED LOOP TRANSFER FUNCTION AND CHARACTERISTIC POLYNOMIAL

The characteristics and properties of the closed loop system play an important role in control studies. Consider the unity feedback system of figure 8. The input is $R(z)$ and the output is denoted by $Y(z)$, $C(z)$ is the digital controller. The goal is to derive the closed loop transfer function. The error signal is given by

$$E(z) = R(z) - Y(z)$$

We also have

$$Y(z) = C(z)G_{za}(z)E(z)$$

By definition, the open loop system is $C(z)G_{za}(z)$. Substituting the error by its value in equation (75), we get

$$Y(z) = C(z)G_{za}(z)(R(z) - Y(z))$$

from which the transfer function is derived

$$G_{cl}(z) = \frac{C(z)G_{za}(z)}{1 + C(z)G_{za}(z)}$$