Discrete Time Signals

- Discrete time systems are more easily analysed using $\mathcal{Z}$ transforms.

- $\mathcal{Z}$ transforms allow the designer to use the same (to a large extent) frequency domain techniques used in the $s$-domain.

- There are 2 ways to approach a $\mathcal{Z}$ transform:
  1) Think in terms of systems that are intrinsically discrete represented by

$$a_m y(kT+mT) + a_{m-1} y(kT+(m-1)T) + \ldots + a_0 y(kT) = 2$$

$$\Leftrightarrow b_l u(kT+lT) + b_{l-1} u(kT+(l-1)T) + \ldots + b_0 u(kT)$$

  2) Another approach is to sample a continuous signal

Assuming a signal $x_c(t)$ is band-limited through the use of an anti-aliasing filter a typical system realization may look like:

- Here the term DSP is reserved for Discrete Time Signal Processing
  - May be implemented in "all digital"
  - May be implemented in "discrete time analog"

- Must pay attention to the signals & their spectra
Consider the following signals

- Assume $s(t)$ is a periodic impulse train in time with period $T$

\[ T = \frac{1}{f_s} \]

where $f_s$ is the sampling frequency

What can be said about the time signals & their spectral?

- $x_s(t)$ & $x_c(t)$ have the same frequency spectrum but the baseband spectrum repeats every $f_s$ for $x_s(t)$

- Here $x(n)$ has the same frequency spectrum as $x_s(t)$ but the sampling frequency is normalized to 1

- Finally the frequency spectrum for $x_{sn}(t)$ equals that of $x_s(t)$ multiplied by $\frac{\sin x}{x}$ attenuating higher frequency Components
The ideal Sampler

- The ideal sampler is a switch that opens & closes instantaneously at every $T$ units of time

\[ x(t) \xrightarrow{T} x^*(t) \]

- No real switch will open & close instantaneously and there will be a finite pulse width
  - We will look at both case

- First consider the case of an ideal impulse
  - The sampled signal takes the form

\[ x^*(t) = S_T(t) x(t) \]

where

\[ S_T(t) \triangleq \sum_{k=0}^{\infty} S(t-kT) \]

The Laplace Transform of $x^*(t)$

- The Dirac delta function is not a normal function but is improper
  - Taking the Laplace of $x^*(t)$ eliminates it altogether

\[ \mathcal{L}\{x^*(t)\} \triangleq X^*(s) \]
\[ x \{ x^*(t) \} = x^*(s) \]
\[ = \sum_{k=0}^{\infty} \int_0^{\infty} x(t) e^{-st} s(t-kT) \, dt \]
\[ = \sum_{k=0}^{\infty} x(kT) e^{-kTs} \]

*In John's & Martin the pulse has a width \( \tau \) and the sampled \( t \) continuous-time signal are related at time \( nt \) as (instead of \( kT \))

\[ x_{sn}(t) = \frac{x_c(nt)}{\tau} [u(t-nt) - u(t-nt-\tau)] \]

*recall that the \( \mathcal{L} \)-aplace transform of \( u(t-\tau) \) is \( \frac{1}{s} e^{-\tau s} \)

- Applying this principle to \( x_{sn}(t) \)

\[ X_{sn}(s) = \left[ \frac{1}{s} e^{-nt} - \frac{1}{s} e^{-nt} e^{-\tau} \right] \frac{x_c(nt)}{\tau} \]
\[ = \frac{1}{\tau} \left( \frac{1-e^{-\tau s}}{s} \right) x_c(nt) e^{-snT} \]

Summing of the contributions of the pulses

\[ X_s(s) = \frac{1}{\tau} \left( 1-e^{-\tau s} \right) \sum_{n=-\infty}^{\infty} x_c(nt) e^{-snT} \]

Now as \( \tau \to 0 \)

\[ \frac{1}{\tau} \left( 1-e^{-\tau s} \right) \bigg|_{\tau \to 0} = 1 \]

*The proof requires use of L'Hopital's rule

- rewriting the equation and taking the limit

\[ \lim_{\tau \to 0} \frac{e^{\tau s} - 1}{\tau s} e^{\tau s} = \frac{1-1}{0(1)} = \frac{0}{0} \]
Taking the derivative

\[
\frac{sTe^{st}(sTe^{st}) - (sTe^{st})(e^{st} - 1)}{(sTe^{st})^2 e^{2st}}
\]

or

\[
\frac{(sTe^{st})^2 e^{2st} - (sTe^{st})^2 e^{2st} + (sTe^{st})^2 e^{st}}{(sTe^{st})^2 e^{2st}}
\]

yields

\[
\lim_{t \to 0} \frac{1}{e^{st}} = 1 \quad \text{Q.E.D.}
\]

which means

\[
X_s(s) = \sum_{n=-\infty}^{\infty} x_c(nT) e^{-snT} \quad \text{which is bilateral vs.}
\]

\[
X^*(s) = \sum_{k=0}^{\infty} x(kT) e^{-ksT} \quad \text{which is unilateral}
\]

Spectra of discrete-time signals

Note that the continuous-time signal \( x_c(\omega) \) is multiplied by an impulse train \( s(t) \) to give \( x_s(t) \) or

\[
x_c(t) \rightarrow \bigotimes s(t) \rightarrow x_s(t)
\]

So \( x_s(t) = x_c(t) s(t) \) where \( s(t) = \sum_{n=-\infty}^{\infty} s(t-nT) \)

Since the Fourier Transform of an impulse train results in another impulse train the spectrum of \( s(t) \rightarrow s(j\omega) \) can be written as

\[
s(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} s(\omega - k\frac{2\pi}{T})
\]
Also, multiplication in the time domain is convolution in the frequency domain so \( x_s(t) = x_c(t) s(t) \) transforms into

\[
x_s(j\omega) = \frac{1}{2\pi} x_c(j\omega) * s(j\omega) \tag{1}
\]

This results in a spectrum

\[
x_s(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} x_c(j\omega - j\frac{k2\pi}{T}) \ldots \tag{11}
\]

or

\[
x_s(f) = \frac{1}{T} \sum_{k=-\infty}^{\infty} x_c(j2\pi f - j\frac{k2\pi}{T} f_s) \ldots \tag{21}
\]

where \( \omega = 2\pi f \) & \( \frac{1}{T} = f_s \)

Note from (11) & (21) above that the spectrum of \( x_s(t) \), or the sampled signal, equals the sum of the shifted spectra of \( x_c(t) \), the continuous-time signal.

- No aliasing will occur if \( x_c(j\omega) \) is band-limited to \( f_s/2 \)

- Recall the graph for the spectrum of \( x_s(f) \) where the baseband spectrum kept repeating

- Now the reason can be seen! For a discrete time signal

\[
x_s(f) = x_s(f \pm k f_s) \text{ where } k \text{ is an arbitrary integer}
\]
From the previous derivation:

\[ X^*(s) = \sum_{k=0}^{\infty} x(kT) e^{-kTs} \]

We wish to substitute for \( s \) ... define

\[ X(Z) = X^*(s) \mid s = \frac{\ln(Z)}{T} \]

Then

\[ X(Z) = \sum_{k=0}^{\infty} x(kT) e^{-kT(\ln(Z)/T)} = \sum_{k=0}^{\infty} x(kT)(e^{\ln(Z)})^{-k} \]

for

\[ X(Z) = \sum_{k=0}^{\infty} x(kT) Z^{-k} \]

In terms of signals in John & Haustin:

\[ X(Z) = \sum_{n=-\infty}^{\infty} x_c(nT) Z^{-n} \]

- Where \( X(Z) \) is the \( Z \)-transform of the samples \( x_c(nT) \)
- \( X(Z) \) is not a function of the sampling rate but is related only to the numbers of \( x_c(nT) \)
- If \( T \) is normalized, as seen by the plots earlier, \( x_5(f) \) & \( x(\omega) \) are related by

\[ x_5(f) = X(\frac{2\pi f}{f_5}) \text{ or } \omega = \frac{2\pi f}{f_5} \]

Example: Continuous-time sinusoid at 1kHz, \( f_5 = 4kHz \) changes by \( \pi/2 \) rads between each sample.
Downsampling & Upsampling

Downsampling → Reduction in sampling rate
Upsampling → Increase in sampling rate

Downsampling

Keep every L^{th} sample and discard the others

• Note how downsampling affected the spectra
  - Downsampled by L = 4 now the spectrum expands by 4
  - Must be sure not to introduce aliasing :: the spectra of
    the original signal must be band-limited to \frac{\pi}{L} prior to
down sampling
  - Signal must then be sampled L times above its minimum
    Sampling rate to not loose information when downsampling

Upsampling

• Upsampling inserts L-1 zero values between samples
  - Upsampled signal spectra are identical to the original signal
    but are re-normalized along the frequency axis
  - When a signal is upsamplied by a factor L the frequency axis is
    Scaled by L
So upsampling by \( L \) has resulted in the axis being scaled by \( L \)

**Sample & hold response**

- Need to change a discrete-time signal back into an analog signal
  - Use a sample & hold circuit
  - What happens to the frequency response when this is done?

A sampled and held signal \( x_{sh}(t) \) is related to its sampled signal by

\[
x_{sh}(t) = \sum_{n=-\infty}^{\infty} x_c(nT)[u(t-nT) - u(t-nT-T)]
\]

Taking the Laplace results in

\[
x_{sh}(s) = \frac{1-e^{-sT}}{s} \sum_{n=-\infty}^{\infty} x_c(nT)e^{-sNT}
\]

\[
= \frac{1-e^{-sT}}{s} x_s(s)
\]

So the "hold" transfer function is (zero order hold)

\[
H_{sh}(s) = \frac{1-e^{-sT}}{s}
\]
The spectrum for \( H_{sh}(s) \) is found by evaluating \( H_{sh}(s) \) for \( s \to j\omega \)

\[
H_{sh}(j\omega) = \frac{1 - e^{-j\omega T}}{j\omega} = T e^{-j\omega T/2} \frac{\sin(\omega T/2)}{(\omega T/2)}
\]

The magnitude response is given by

\[
|H_{sh}(j\omega)| = T \frac{\left|\sin(\omega T/2)\right|}{\omega T/2}
\]

Alternatively

\[
|H_{sh}(f)| = T \frac{\left|\sin(\pi f/ f_s)\right|}{\pi f/ f_s}
\]

The results for this spectrum were shown graphically earlier.

- Note how the frequency was "shaped" by the sinc signal.

- Looking at the original signal & the sampled & held signal before the A/D...

\[
\begin{align*}
X_c(t) &\rightarrow S/H \rightarrow X_{sh}(t) \rightarrow A/D \rightarrow X(n) = X_c(nT) \\
\end{align*}
\]

... The S/H causes \( X_{sh}(t) \) to have smaller images at high freq. due to the sinc response.

- The images of \( X(n) \) however are all the same height since \( X(n) \) is a discrete-time signal.

- Therefore the S/H does not aid in anti-aliasing requirement.