ORTHONORMALIZATION

- $X$ IS NORMALIZED IF $\|X\|_2 = 1$ OR EQUIV. $X^\top X = 1$

- $x_i, x_j$ ARE ORTHOGONAL IF $x_i^\top x_j = 0$
  AND ORTHONORMAL IF $\|x_i\|_2 = 1$ AND $\|x_j\|_2 = 1$

- ANY LINEAR SET OF INDEPENDENT VECTORS CAN BE MADE INTO AN ORTHONORMAL SET.

- THE "SCHMIDT ORTHONORMALIZATION PROCEDURE" IS ONE SUCH PROCEDURE.

- GIVEN $A \Rightarrow A_{\text{on}}$ (ORTHONORMALIZED)

  $A_{\text{GRAMIAN}} = A_{\text{on}}^\top A_{\text{on}} = I_{\text{mxm}}$ (MxM IDENTITY MATRIX)

  (Note: Gramian of $A = A^\top A \neq I_{\text{mxm}}$)

- IF $\det A_{\text{GRAMIAN}} = \det |A^\top A| \neq 0$, THE SET OF VECTORS THAT MAKE UP $A$
  ARE LINEARLY INDEPENDENT AND CAN BE USED AS A BASIS WHERE $\text{DIM (RANGE)} = \text{DIM (SPACE)}$
  = NUMBER OF BASIS VECTORS (COORDINATES).
orthnormex
A =
 1  1  1
 1  2  3
 1  3  2

rankA= rank(A)
rankA = 3

Gramian_A= A'*A
Gramian_A =
 3   6   6
 6  14  13
 6  13  14

detGrmA= det(Gramian_A)
detGrmA = 9

Aon= orthonormal(A)  (A FUNCTION CREATED BY PROF. HELLER)
Aon =
 0.5774 -0.7071 -0.4082
 0.5774  0.0000  0.8165
 0.5774  0.7071 -0.4082
% orthonormal.m, Prof. M. Heller
% function aon= orthonormal(a)
% Schmidt Orthonormalization of a matrix
function aon= orthonormal(a)
[r,c]=size(a);
aon=[]; qpast=zeros(r,1); sumqeq=zeros(r,1);
for j= 1:c
    e= a(:,j);
    sumqeq= sumqeq + (qpast'*e).*qpast;
    u= e - sumqeq;
    q= u./norm(u,2);
    aon(1:r,j)= q;
    qpast=q;
end

Gramian_Aon= Aon'*Aon
Gramian_Aon =
 1.0000  0.0000  0.0000
 0.0000  1.0000 -0.0000
 0.0000 -0.0000  1.0000

Gramian_Aon2= Aon2'*Aon2
Gramian_Aon2 =
 1.0000 -0.0000 -0.0000
-0.0000  1.0000 -0.0000
-0.0000 -0.0000  1.0000
LINEAR ALGEBRAIC EQUATIONS

Given \( Ax = y \), \( A \in \mathbb{R}^{m \times n}, \ x \in \mathbb{R}^{n}, \ y \in \mathbb{R}^{m} \)

We have \( m \) equations and \( n \) unknowns.

Definitions

1. If \( Ax = 0 \), \( x \) is a "null vector" of \( A \)
2. "Null space" contains all null vectors.
3. \( \text{nullity}(A) = \# \text{Col} \setminus \text{rank}(A) \).

Theorem 3.1

Given \( Ax = y \), a solution \( x \) exists iff \( \text{rank}(A) = m \) (full row rank).

Theorem 3.2

Given \( Ax = y \), the solution \( x \) is unique, iff \( \text{rank}(A) = n \) (full column rank).

If \( \text{nullity}(A) = m - \text{rank}(A) > 0 \), then there are an \( \infty \) number of solutions not unique

\[ x = x^0 + N\alpha \] where \( \alpha \) is any vector, \( x^0 \in \mathbb{R}^n \), and \( N \) is a basis matrix of the null space of \( A \) and \( x^0 \) is a solution. Also,

\[ A(x - x^0) = 0 \Rightarrow (x - x^0) \text{ is in the null space} \]
A =
  0  1  1  2
  1  2  3  4
  2  0  2  0

[m,n] = size(A)
m = 3
n = 4

rankA = rank(A)
rankA = 2

Gramian_A = A'*A
Gramian_A =
  5  2  7  4
  2  5  7 10
  7  7 14 14
  4 10 14 20

detGrmA = det(Gramian_A)
detGrmA = 0

N = null(A)
N =
  0.4234 -0.5247
  0.7808  0.4567
 -0.4234  0.5247
 -0.1787 -0.4907

nullityA = n - rankA
nullityA = 2

y = [-4 -8 0]'
y =
  -4
  -8
   0

xs2 = A\y

Warning: Rank deficient, rank = 2  tol =  3.9721e-015.
> In C:\MDH_Toolbox\EEE241\example3_3.m at line 14
\[ \text{xs2} = \\
0 \\
0 \\
0 \\
-2 \]

\[ A^{*}\text{xs2} \]
\[ \text{ans} = \\
-4 \\
-8 \\
0 \]

\[ \text{xs} = \text{pinv}(A)^{*}y \]
\[ \text{xs} = \\
0.3636 \\
-0.7273 \\
-0.3636 \\
-1.4545 \]

\[ A^{*}\text{xs} \]
\[ \text{ans} = \\
-4.0000 \\
-8.0000 \\
0.0000 \]

\[ \text{alpha} = \text{ones} (\text{nullity}A,1) \ % [1;1] \]
\[ \text{alpha} = \\
1 \\
1 \]

\[ \text{xp} = \text{xs} - N^{*}\text{alpha} \]
\[ \text{xp} = \\
0.4650 \\
-1.9648 \\
-0.4650 \\
-0.7851 \]

\% Text solution
\[ \text{xp2} = [0 \ -4 \ 0 \ 0]' \]
\[ A^{*}\text{xp2} \]
\[ \text{ans} = \\
-4 \\
-8 \\
0 \]
% Find an alpha set
% Xs = Xp2 + N*alpha2
alpha2 = pinv(N)*(xs - xp2)
alpha2 =
    3.1230
    1.8269

Xs = xp2 + N*alpha2
\% CHECK
Xs =
    0.3636
   -0.7273
   -0.3636
  -1.4545

echo off
»