Example 3.10 Consider the Jordan block of order 4:

\[
\hat{A} = \begin{bmatrix}
\lambda_1 & 1 & 0 & 0 \\
0 & \lambda_1 & 1 & 0 \\
0 & 0 & \lambda_1 & 1 \\
0 & 0 & 0 & \lambda_1 \\
\end{bmatrix}
\] (3.46)

Its characteristic polynomial is \((\lambda - \lambda_1)^4\). Although we can select \(h(\lambda)\) as \(\beta_0 + \beta_1 \lambda + \beta_2 \lambda^2 + \beta_3 \lambda^3\), it is computationally simpler to select \(h(\lambda)\) as

\[
h(\lambda) = \beta_0 + \beta_1 (\lambda - \lambda_1) + \beta_2 (\lambda - \lambda_1)^2 + \beta_3 (\lambda - \lambda_1)^3
\]

This selection is permitted because \(h(\lambda)\) has degree \((n - 1) = 3\) and \(n = 4\) independent unknowns. The condition \(f(\lambda) = h(\lambda)\) on the spectrum of \(\hat{A}\) yields immediately

\[
\beta_0 = f(\lambda_1), \quad \beta_1 = f'(\lambda_1), \quad \beta_2 = \frac{f''(\lambda_1)}{2!}, \quad \beta_3 = \frac{f^{(3)}(\lambda_1)}{3!}
\]

Thus we have

\[
f(\hat{A}) = f(\lambda_1)I + \frac{f'(\lambda_1)}{1!} (\hat{A} - \lambda_1 I) + \frac{f''(\lambda_1)}{2!} (\hat{A} - \lambda_1 I)^2 + \frac{f^{(3)}(\lambda_1)}{3!} (\hat{A} - \lambda_1 I)^3
\]

Using the special forms of \((\hat{A} - \lambda_1 I)^k\) as discussed in (3.40), we can readily obtain

\[
f(\hat{A}) = \begin{bmatrix}
0 & f(\lambda_1) & f'(\lambda_1)/1! & f''(\lambda_1)/2! & f^{(3)}(\lambda_1)/3! \\
0 & 0 & f(\lambda_1) & f'(\lambda_1)/1! & f''(\lambda_1)/2! \\
0 & 0 & 0 & f(\lambda_1) & f'(\lambda_1)/1! \\
0 & 0 & 0 & 0 & f(\lambda_1) \\
\end{bmatrix}
\] (3.47)

If \(f(\lambda) = e^{\lambda t}\), then

\[
e^{\hat{A}t} = \begin{bmatrix}
e^{\lambda t} & te^{\lambda t} & t^2e^{\lambda t}/2! & t^3e^{\lambda t}/3! \\
0 & e^{\lambda t} & te^{\lambda t} & t^2e^{\lambda t}/2! \\
0 & 0 & e^{\lambda t} & te^{\lambda t} \\
0 & 0 & 0 & e^{\lambda t} \\
\end{bmatrix}
\] (3.48)

Because functions of \(A\) are defined through polynomials of \(A\), Equations (3.41) and (3.42) are applicable to functions.
A = diag(ones(4,1),1); A = zeros(5,5)+A; sym r1 t A = sym(A); A = r1*eye(5) + A
A =
[ r1, 1, 0, 0, 0]
[ 0, r1, 1, 0, 0]
[ 0, 0, r1, 1, 0]
[ 0, 0, 0, r1, 1]
[ 0, 0, 0, 0, r1]
expAt = expm(A.*t)
expAt =
[exp(r1.*t), t*exp(r1.*t), 1/2*t^2*exp(r1.*t), 1/6*t^3*exp(r1.*t), 1/24*t^4*exp(r1.*t)]
[ 0, exp(r1.*t), t*exp(r1.*t), 1/2*t^2*exp(r1.*t), 1/6*t^3*exp(r1.*t)]
[ 0, 0, exp(r1.*t), t*exp(r1.*t), 1/2*t^2*exp(r1.*t)]
[ 0, 0, 0, exp(r1.*t), t*exp(r1.*t)]
[ 0, 0, 0, 0, exp(r1.*t)]

B = diag(ones(4,1),1); B = zeros(5,5)+B;
B(4,5)=0; B(3,4)=0;
syms r1 t B = sym(B);
B = r1*eye(5) + B
B =
[ r1, 1, 0, 0, 0]
[ 0, r1, 1, 0, 0]
[ 0, 0, r1, 0, 0]
[ 0, 0, 0, 0, r1]
[ 0, 0, 0, 0, r1]
expBt = expm(B.*t)
expBt =
[exp(r1.*t), t*exp(r1.*t), 1/2*t^2*exp(r1.*t), t*exp(r1.*t), 0, 0]
[ 0, exp(r1.*t), t*exp(r1.*t), t*exp(r1.*t), 0, 0]
[ 0, 0, exp(r1.*t), t*exp(r1.*t), 0, 0]
[ 0, 0, 0, exp(r1.*t), t*exp(r1.*t), 0, 0]
[ 0, 0, 0, 0, exp(r1.*t), t*exp(r1.*t), 0, 0]

C = diag(ones(4,1),1);
C = zeros(5,5)+C;
C(2,3)=0; C(4,5)=0;
syms r1 t C = sym(C);
C = r1*eye(5) + C
C =
[ r1, 1, 0, 0, 0]
[ 0, r1, 1, 0, 0]
[ 0, 0, r1, 1, 0]
[ 0, 0, 0, r1, 0]
[ 0, 0, 0, 0, r1]
expCt = expm(C.*t)
expCt =
[exp(r1.*t), t*exp(r1.*t), 0, 0, 0]
[ 0, exp(r1.*t), 0, 0, 0]
[ 0, 0, exp(r1.*t), t*exp(r1.*t), 0]
[ 0, 0, 0, exp(r1.*t), 0]
[ 0, 0, 0, 0, exp(r1.*t)]
Example 3.11 Consider

\[
A = \begin{bmatrix}
\lambda_1 & 1 & 0 & 0 & 0 \\
0 & \lambda_1 & 1 & 0 & 0 \\
0 & 0 & \lambda_1 & 0 & 0 \\
0 & 0 & 0 & \lambda_2 & 1 \\
0 & 0 & 0 & 0 & \lambda_2
\end{bmatrix}
\]

\[f(x) = (s - x)^{-1}, \quad F(A) = ?\]

Because \(A\) is in Jordan form:

\[
F(A) = (sI - A)^{-1} = \begin{bmatrix}
\frac{1}{s - \lambda_1} & \frac{1}{(s - \lambda_1)^2} & \frac{1}{(s - \lambda_1)^3} & 0 & 0 \\
0 & \frac{1}{s - \lambda_1} & \frac{1}{(s - \lambda_1)^2} & 0 & 0 \\
0 & 0 & \frac{1}{s - \lambda_1} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{s - \lambda_2} & \frac{1}{(s - \lambda_2)^2} \\
0 & 0 & 0 & 0 & \frac{1}{s - \lambda_2}
\end{bmatrix}
\]

\[f'(x) \quad f''(x) \quad f'''(x) \quad f''''(x) \]

Notice that the main diagonal entries are \(f'(x)\).
**Exponential Matrix**

$e^{At}$ is very important because it gives us a closed form solution to all LTI systems.

**The Taylor Series Expansion of $e^{At}$**

$$e^{At} = 1 + At + \frac{A^2 t^2}{2} + \ldots + \frac{A^m t^m}{m!} + \ldots$$

Converges for all finite $t$ and $A$; hence

$$e^{At} = I + tA + \frac{A^2 t^2}{2} + \ldots = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$$