\[ \nabla^2 f(x,y) \equiv \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \text{ (ISOTROPIC)} \]

The above definition implies that the Laplacian is "ISOTROPIC," i.e., it is directionally invariant (can't determine direction from result).

However, we can obtain a direction by using two WOPRs as we did in obtaining the gradient.

First we approximate the 2nd derivative in the x dir:

\[ \nabla^2 f_x(x,y) \approx \nabla (\nabla f_x(x,y)) = \nabla \left( \frac{f(x+\Delta x,y) - f(x,y)}{\Delta x} \right) = \frac{1}{\Delta x} \left( \frac{f(x+\Delta x,y) - f(x,y)}{\Delta x} \right) \]

\[ \nabla^2 f_x(x,y) \approx \frac{f(x+\Delta x,y) - 2f(x,y) + f(x-\Delta x,y)}{\Delta x^2} \rightarrow \]

Similarly,

\[ \nabla^2 f_y(x,y) \approx \frac{f(x,y+\Delta y) - 2f(x,y) + f(x,y-\Delta y)}{\Delta y^2} \rightarrow \]

Hence, the \textbf{edge direction} \( \alpha \) is:

\[ \alpha = \tan^{-1} \left( \frac{f_x}{f_y} \right) \]

\[ \nabla^2 f(x,y) \approx \nabla^2 f_x + \nabla^2 f_y = \nabla^2 \phi \]

Notice that when converting the 1x3x3x1 WOPR to 3x3:

\[ \begin{bmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} \]

A commonly used 3x3 isotropic Laplacian

\[ \begin{bmatrix} +1 & +1 & +1 \\ +1 & -8 & +1 \\ +1 & +1 & +1 \end{bmatrix} \]

Scale factor = \( \frac{1}{4} \)

And the one on the next page is used to enhance the edge. Side note: A 5x5 WOPR is equivalent to two 3x3 WOPRs. Which approach is computationally more efficient?
Another way of looking at this filter is to say that it has the effect of sharpening edges by pushing a pixel away from its neighbors. To see this, consider that the above masks reduce to \((-1 \ 3 \ -1)\) in one dimension, and thus the filter may be written as:

\[ v_i + (v_i - v_{i-1}) + (v_i - v_{i+1}) \]

This is illustrated in Figure 41.

Figure 41. The Laplacian as pushing a pixel away from its neighbors.
The Laplacian-plus-original filter is analogous to the photographic process of "unsharp masking." In this process, a film is exposed through a negative superimposed on a slightly defocused positive transparency, thus subtracting the local mean, and the result is an image with improved edges.

The (negative) Laplacian of a function may be represented as the function minus its local mean.

**AN INTRODUCTION TO DIGITAL IMAGE PROCESSING**
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![Diagram of image sharpening](image)

**Figure 42. Image sharpening:** The first figures show plots of \( f, f', \) \( f'' = \nabla^2 f, \) and \( f - \nabla^2 f. \) Also shown are plots of \( f \) and \( f + (f - f) \), which is a form of Unsharp Masking.

Improved edges
FIGURE 4.21 2nd derivatives, direction and zero crossings. Spatial and directional factors interact in the definition of a zero-crossing segment: (a) shows an intensity change, and (b), (c), and (d) show values of the second directional derivative near the origin at various orientations across the change. In (b), the derivative is taken parallel to the x-axis, and in (c), and (d), at 30° and 60° to it. There is zero-crossing at every orientation except for \( d^2I/dy^2 \), which is identically zero. Since the zero-crossings line up along the x-axis, this is the direction that is chosen. In this example, it is also the direction that maximizes the slope of the second derivative (from [Marr80]). © 1980, the Royal Society.

FIGURE 4.20 Discrete Laplacian for edge enhancement
(a) Input image
(b) Edge enhanced image using Laplacian kernel