Design via State Space

SOLUTION TO CASE STUDY CHALLENGE

Antenna Control: Design of Controller and Observer

a. We first draw the signal-flow diagram of the plant using the physical variables of the system as state variables.

\[
\begin{align*}
\frac{1}{s} & \quad z_3 & 0.8 & \quad \frac{1}{s} & \quad z_2 & \quad \frac{1}{s} & \quad z_1 & 0.2 & \quad y \\
3.18 & \quad u & 2000 & \quad \text{100} & \quad -1.32 & \quad 1 & \quad 0 & \quad -100 & \quad -1.32
\end{align*}
\]

Writing the state equations for the physical variables shown in the signal-flow diagram, we obtain

\[
\dot{z} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1.32 & 0.8 \\ 0 & 0 & -100 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ 2000 \end{bmatrix} u \quad ; \quad y = \begin{bmatrix} 0.2 & 0 & 0 \end{bmatrix} z
\]

The characteristic polynomial for this system is \( s^3 + 101.32s^2 + 132s + 0 \). Hence, the A and B matrices of the phase-variable form are

\[
\begin{align*}
\text{Ax} & \quad 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -132 & -101.32 \\
\text{Bx} & \quad 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -101.32 & 1
\end{align*}
\]

Writing the controllability matrices and their determinants for both systems yields...
The controllability matrices are given by

\[
\begin{array}{ccc|ccc}
0 & 0 & 1600 & 0 & 0 & 1 \\
0 & 1600 & -162112 & 0 & 1 & -101.32 \\
2000 & -200000 & 20000000 & 1 & -101.32 & 10133.7424 \\
\end{array}
\]

The determinant of these matrices are:

\[
\begin{align*}
\text{Det}(CM_z) & = -5.12 \times 10^9 \\
\text{Det}(CM_x) & = -1
\end{align*}
\]

where the system is controllable. Using Eq. (12.39), we find the transformation matrix and its inverse to be

\[
P \quad \text{Transformation Matrix } z = Px
\]

\[
\begin{array}{cccccc}
1600 & 0 & 0 & 0.000625 & 0 & 0 \\
0 & 1600 & 0 & 0 & 0.000625 & 0 \\
0 & 2640 & 2000 & 0 & -0.000825 & 0.0005 \\
\end{array}
\]

The characteristic polynomial of the phase-variable system with state feedback is

\[
s^3 + (k_3 + 101.32)s^2 + (k_2 + 132)s + (k_1 + 0)
\]

For 15% overshoot, \(T_s = 2\) seconds, and a third pole 10 times further from the imaginary axis than the dominant poles, the characteristic polynomial is

\[
(s + 20)(s^2 + 4s + 14.969) = s^3 + 24s^2 + 94.969s + 299.38
\]

Equating coefficients, the controller for the phase-variable system is

\[
K_x \quad \text{Controller for } x \\
299.38 \quad -37.031 \quad -77.32
\]

Using Eq. (12.42), the controller for the original system is

\[
K_z \quad \text{Controller for } z \\
0.1871125 \quad 0.04064463 \quad -0.03866
\]

b. Using \(K_z\), gain from \(\theta_m = -0.1871125\) (including gear train, pot, and operational amplifier); gain from tachometer = -0.04064463; and gain from power amplifier output = 0.03866.
c. Using the original system from part (a) and its characteristic polynomial, we find the observer canonical form which has the following $A$ and $C$ matrices:

$$
\begin{align*}
A &= \begin{bmatrix}
-101.32 & 1 & 0 \\
-132 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix}, \\
C &= \begin{bmatrix}
1 & 0 \\
\end{bmatrix}
\end{align*}
$$

Writing the observability matrices and their determinants for both systems yields

$$
\begin{align*}
\text{OM}_z & : \text{Observability Matrix of } z \\
\text{OM}_x & : \text{Observability Matrix of } x \\
\text{Det}(\text{OM}_z) & = 0.0064 \\
\text{Det}(\text{OM}_x) & = 1
\end{align*}
$$

where the system is observable. Using Eq. (12.89), we find the transformation matrix and its inverse to be

$$
\begin{align*}
P & = \begin{bmatrix}
5 & 0 & 0 \\
-506.6 & 5 & 0 \\
62500 & -625 & 6.25 \\
\end{bmatrix}, \\
\text{PINV} & = \begin{bmatrix}
0.20 & 0.00 & 0.00 \\
20.26 & 0.20 & 0.00 \\
26.40 & 20.00 & 0.16 \\
\end{bmatrix}
\end{align*}
$$

The characteristic polynomial of the dual phase-variable system with state feedback is
For 10% overshoot, \( \omega_n = 10\sqrt{14.969} = 38.69 \text{ rad/s} \), and a third pole 10 times further from the imaginary axis than the dominant observer poles, the characteristic polynomial is

\[
(s + 228.72)(s^2 + 45.743s + 1496.916) = s^3 + 274.46s^2 + 11959s + 3.4237 \times 10^5
\]

Equating coefficients, the observer for the observer canonical system is

\[
\begin{align*}
L_x & \quad \text{Observer for } x \\
173.14 & \\
11827 & \\
342370 & 
\end{align*}
\]

Using Eq. (12.92), the observer for the original system is

\[
\begin{align*}
L_z & \quad \text{Observer for } z \\
865.7 & \\
-28577.724 & \\
5569187.5 & 
\end{align*}
\]

d.
e.

Program:

'Controller'
A=[0 1 0;0 -1.32 0.8;0 0 -100];
B=[0;0;2000];
C=[0.2 0 0];
D=0;
pos=input('Type desired %OS');
Ts=input('Type desired settling time');
z=(-log(pos/100))/(sqrt(pi^2+log(pos/100)^2));
wn=4/(z*Ts); %Calculate required natural frequency.
[num,den]=ord2(wn,z); %Produce a second-order system that meets the transient response requirements.
r=roots(den); %Use denominator to specify dominant poles.
poles=[r(1) r(2) 10*real(r(1))]; %Specify pole placement for all poles.
K=acker(A,B,poles)

'Observer'
pos=input('Type desired %OS');
z=(-log(pos/100))/(sqrt(pi^2+log(pos/100)^2));
wn=10*wn %Calculate required natural frequency.
[num,den]=ord2(wn,z); %Produce a second-order system that meets the transient response requirements.
r=roots(den); %Use denominator to specify dominant poles.
poles=[r(1) r(2) 10*real(r(1))]; %Specify pole placement for all poles.
l=acker(A',C',poles)'

Computer response:
ans =
Controller
Type desired %OS 15
Type desired settling time 2
K =
0.1871 0.0406 -0.0387
ans =
Observer
Type desired %OS 10
wn =
38.6899
l =
1.0e+006 *
0.0009
-0.0286
5.5691
ANSWERS TO REVIEW QUESTIONS

1. Both dominant and non-dominant poles can be specified with state-space design techniques.
2. Feedback all state variables to the plant's input through a variable gain for each. Decide upon a closed-loop characteristic equation that has a pole configuration to yield a desired response. Write the characteristic equation of the actual system. Match coefficients and solve for the values of the variable gains.
3. Phase-variable form
4. The control signal developed by the controller must be able to affect every state variable.
5. If the signal-flow diagram is in the parallel form, which leads to a diagonal system matrix, controllability can be determined by inspection by seeing that all state variables are fed by the control signal.
6. The system is controllable if the determinant of the controllability matrix is non-zero.
7. An observer is a system that estimates the state variables using information from the output of the actual plant.
8. If the plant's state-variables are not accessible, or too expensive to monitor
9. An observer is a copy of the plant. The difference between the plant's output and the observer's output is fed back to each of the derivatives of the observer's state variables through separate variable gains.
10. Dual phase-variable
11. The characteristic equation of the observer is derived and compared to a desired characteristic equation whose roots are poles that represent the desired transient response. The variable gains of each feedback path are evaluated to make the coefficients of the observer's characteristic equation equal the coefficients of the desired characteristic equation.
12. Typically, the transient response of the observer is designed to be much faster than that of the controller. Since the observer emulates the plant, we want the observer to estimate the plant's states rapidly.
13. Det[A - BK], where A is the system matrix, B is the input coupling matrix, and K is the controller.
14. Det[A - LC], where A is the system matrix, C is the output coupling matrix, and L is the observer.
15. The output signal of the system must be controlled by every state variable.
16. If the signal-flow diagram is in the parallel form, which leads to a diagonal system matrix, observability can be determined by inspection by seeing that all state variables feed the output.
17. The system is observable if the determinant of the observability matrix is non-zero.
\[ d_i = a_i + k_{n-i} ; \quad i = 0, 1, 2, \ldots, n-1 \]

from which

\[ k_{n-i} = d_i - a_i \]  \hspace{1cm} (1)

b. The desired characteristic equation is

\[ s^3 + 15.9s^2 + 136.08s + 413.1 = 0 \]

the characteristic equation of the plant is

\[ s^3 + 5s^2 + 4s + 0 = 0 \]

Using Eq. (1) above, \( k_{3-i} = d_i - a_i \). Therefore, \( k_3 = d_0 - a_0 = 413.1 - 0 = 413.1; \)
\( k_2 = d_1 - a_1 = 136.08 - 4 = 132.08; \)
\( k_1 = d_2 - a_2 = 15.9 - 5 = 10.9 \). Hence,

\[ K = \begin{bmatrix} 10.9 & 132.08 & 413.1 \end{bmatrix} \]

6. Using Eqs. (4.39) and (4.34) to find \( \zeta = 0.5169 \) and \( \omega_n = 7.3399 \), respectively. Factoring the denominator of Eq. (4.22), the required poles are \(-3.7942 \pm j6.2832\). We place the third pole at -2 to cancel the open loop zero. Multiplying the three closed-loop pole terms yields the desired characteristic equation:

\[ s^3 + 9.5885s^2 + 69.0516s + 107.7493 = 0 \]

Since \( G(s) = \frac{100s^2 + 2200s + 4000}{s^3 + 8s^2 + 19s + 12} \), the controller canonical form is

\[ A = \begin{bmatrix} -8 & -19 & -12 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad C = \begin{bmatrix} 100 & 2200 & 4000 \end{bmatrix} \]

contains the coefficients of the characteristic equation. Thus comparing the first row of \( A \) to the desired characteristic equation and using the results of Problem 5, \( k_1 = -(9.5885 - 8) = 1.5885; \)
\( k_2 = -(69.0516 - 19) = 50.0516; \) and \( k_3 = -(107.7493 - 12) = 95.7493. \)

7. The plant is given by

\[ G(s) = \frac{20(s + 2)}{s(s + 4)(s + 6)} = \frac{20s + 40}{s^3 + 10s^2 + 24s + 0} \]

The characteristic polynomial for the plant with phase-variable state feedback is

\[ s^3 + (k_3 + 10)s^2 + (k_2 + 24)s + (k_3 + 0) \]

The desired characteristic equation is

\[ (s + 20)(s^2 + 4s + 11.45) = s^3 + 24s^2 + 91.45s + 229 \]

based upon 10% overshoot, \( T_s = 2 \) seconds, and a third pole ten times further from the imaginary axis than the dominant poles. Comparing the two characteristic equations,

\[ k_1 = 229, \quad k_2 = 67.45, \quad \text{and} \quad k_3 = 14. \]