Chapter 8

Root Locus Techniques

- Closed-loop poles using KGH and root-locus
- How to sketch root-locus
- Controller design using root-locus
Figure 8.1

a. Closed-loop system; b. equivalent transfer function

\[ G_{cl}(s) = \frac{K N_{g} N_{h}}{D_{g}(s) D_{h}(s) + K N_{g} N_{h}(\Delta \tau)} \]

\[ G_{H}(s) = \frac{N_{g}(s)}{D_{g}(s)} \]

\[ H_{e}(s) = \frac{N_{h}(s)}{D_{h}(s)} \]

\[ \tau(s) = G_{cl}(s) \]

\[ \text{Poles of } \tau(s) = \text{Poles of } G_{H} \]

\[ K = 0, \quad \Delta \tau = \Delta_{e} D_{h} = 0, \quad \text{Poles of } \tau(s) = \text{Poles of } G_{H} \]

\[ K = \infty, \quad \Delta \tau = \frac{D_{g} D_{h} + N_{g} N_{h}}{K} = 0, \quad \Delta \tau = N_{g} N_{h} = 0 \]

\[ \text{Poles of } \tau(s) = \text{Zeros of } G_{H} \]

©2000, John Wiley & Sons, Inc.
Nise/Control Systems Engineering, 3/e
A root locus is a plot of the poles of \( T(s) \) as \( K \) varies from 0 to \( \infty \).

Starting at \( K = 0 \), the poles of \( GH \) are the poles of \( T(s) \).

At \( K = \infty \), the poles of \( T(s) \) have the same values of the zeros of \( GH \). That is, as \( K \) goes to \( \infty \), the poles of \( T(s) \) move towards the zeros of \( GH \).

Hence, for any gain \( K \), we can find the pole locations of \( T(s) \) from the root locus. However, the most useful properties of the root locus are that 1) we can see how the poles of \( T(s) \) move as \( K \) varies \( K \), 2) we can find \( K \) given the desired pole positions of \( T(s) \), and 3) we can determine the maximum \( K \) value, if any, for asymptotic stability.

We only need to know the factors of \( GH \). The number of zeros of \( GH \) must equal the number of poles of \( GH \) and vice versa.

**Examples**

\[ GH = \frac{s+1}{s(s+2)} \]

\( p_1 = 0, p_2 = -2, z_1 = -1, z_2 = +\infty \) or \( +\infty \)

\[ GH = \frac{s^2}{s+1} \]

\( z_1 = 0, z_2 = 0, p_1 = -1, p_2 = -\infty \) or \( -\infty \)
Simple Root Locus Example

\[ \frac{C}{R} = \frac{KG}{1 + KGH} = \frac{\frac{K(s+1)}{s(s+2)}}{1 + \frac{K(s+1)}{s(s+2)}} = \frac{K(s+1)}{s^2 + (2+k)s + k} \]

Closed-loop poles are found from

\[ \begin{align*}
\Delta_p &= s^2 + (2+k)s + k = 0 \\
\mathcal{P}_1 &= -\frac{(2+k)}{2} + \frac{\sqrt{(2+k)^2 - 4k}}{2} \\
\mathcal{P}_2 &= -\frac{(2+k)}{2} - \frac{\sqrt{(2+k)^2 - 4k}}{2} \\
\end{align*} \]

\( K = 0, \quad \mathcal{P}_1 = -1 + 1 = 0, \quad \mathcal{P}_2 = -1 - 1 = -2 \) (poles of GH)

\( K = \infty \)

\[ \Delta_p = \lim_{K \to \infty} \frac{s^2 + 2s + s + 1}{K} = s + 1 = 0 \]

\( \mathcal{P}_1 = -1, \quad \mathcal{P}_2 = -\infty \) (zeros of GH)

\( K : 0 \to \infty, \quad \mathcal{P}_1 : 0 \to -1, \quad \mathcal{P}_2 : -2 \to -\infty \)

Root locus

Zeros of GH = poles GH

Closed-loop poles of GH for some value of K

Closed-loop poles of GH for some value of K

We will also learn how to find K to achieve the desired response.
Any point on the root-locus (paths of the closed-loop poles) must satisfy

\[ 1 + KGH = 0 \]

or

\[ KGH = -1 \]

or

\[ KGH = 1 e^{-\frac{\theta}{2}(2R+1)180^\circ} \quad R = 0, 1, 2, \ldots \]

or equivalently

\[ \left| KGH \right| = 1 \quad \text{and} \quad \angle KGH = (2R+1)180^\circ, \quad K \geq 0 \]

(Note: \( \angle KGH = \angle GH \))

Testing to see if a point is on the root-locus:

1. Select a point (potential closed-loop pole, \( p = 0.7 + 36 \))

And check to see if \( \theta = \frac{\angle GH}{\angle \theta} = \frac{5}{(2R+1)180^\circ} \)

If \( s = p \) is not on the root-locus, try again!

2. If \( s = p \) is on the root-locus, then we can find \( K \):

\[ K = \frac{1}{\text{Magnitude of } \frac{1}{GH} \big|_{s=p}} = \frac{1}{MGH_{s=p}} \]

For positive feedback systems (\( K > 0 \)), the root-locus is called a "complementary locus." (Skip section 8.9: Root-locus for positive-feedback systems.)
Figure 8.3
Vector representation of Eq. (8.7)

$$\theta_{GH} = \sum_{i=1}^{m} \theta_i^0 - \sum_{j=1}^{n} \theta_j^p$$

$$M_{GH} = \frac{\prod_{i=1}^{m}(\sigma + \alpha_i^0)}{\prod_{j=1}^{n}(\sigma + \alpha_j^p)}$$

Where $s = p$ is the test point (possible closed-loop pole).

$M = \text{number of zeros of } G_H$ and $M = \text{number of poles of } G_H$. 

$K G_H = \frac{K(s+1)}{s(s+2)}$

Test Point

$s = p = -3 + 4j$ (on the root-locus? In this case no)

$s$-plane

Root-locus branches

©2000, John Wiley & Sons, Inc.
Even though the test point \( s = p = -3 + j4 \) is not on the root-locus, we will go ahead and find \( \theta_{GH} = -3 + j4 \) and \( M_{GH} = -3 + j4 \).

\[
K_{GH} = \frac{K(5+1)}{s(5+2)}
\]

\[
\theta_1 = p+1 = (-3+j4)+1 = -2 + j4
\]

\[
\theta_1 = \tan^{-1}(\frac{4}{-2}) = 116.56 \degree
\]

\[
M_1 = \sqrt{(-2)^2 + 4^2} = \sqrt{20} = 4.47
\]

\[
\theta_2 = (p+0) = (-3 + j4) + 0 = -3 + j4
\]

\[
\theta_2 = \tan^{-1}(\frac{4}{-3}) = 126.87 \degree
\]

\[
M_2 = \sqrt{(-3)^2 + 4^2} = 5
\]

\[
\theta_{GH} = (116.56 \degree) - (126.87 \degree + 104.04 \degree) = -114.35 \degree
\]

\([-114.35 \degree \pm (2R+1)180 \degree = 180 \degree \pm -180 \degree , 540 \degree\]

No! Therefore, \( p = -3 + j4 \) is not on the root-locus.

\[
M_{GH} = \frac{4.47}{(5)(4.12)} = 0.217
\]

Since \( p = -3 + j4 \) is not on the root-locus,

\[
K \neq \frac{1}{M_{GH}}
\]
Figure 8.4

a. CameraMan®
Presenter Camera System automatically follows a subject who wears infrared sensors on their front and back (the front sensor is also a microphone); tracking commands and audio are relayed to CameraMan via a radio frequency link from a unit worn by the subject.

b. block diagram.

c. closed-loop transfer function.
Figure 8.5

a. Pole plot from Table 8.1;

b. root locus

<table>
<thead>
<tr>
<th>$K$</th>
<th>CLOSED LOOP Pole 1</th>
<th>CLOSED LOOP Pole 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-10</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>-9.47</td>
<td>-0.53</td>
</tr>
<tr>
<td>10</td>
<td>-8.87</td>
<td>-1.13</td>
</tr>
<tr>
<td>15</td>
<td>-8.16</td>
<td>-1.84</td>
</tr>
<tr>
<td>20</td>
<td>-7.24</td>
<td>-2.76</td>
</tr>
<tr>
<td>25</td>
<td>-5</td>
<td>-5</td>
</tr>
<tr>
<td>30</td>
<td>-5 + j2.24</td>
<td>-5 - j2.24</td>
</tr>
<tr>
<td>35</td>
<td>-5 + j3.16</td>
<td>-5 - j3.16</td>
</tr>
<tr>
<td>40</td>
<td>-5 + j3.87</td>
<td>-5 - j3.87</td>
</tr>
<tr>
<td>45</td>
<td>-5 + j4.47</td>
<td>-5 - j4.47</td>
</tr>
<tr>
<td>50</td>
<td>-5 + j5</td>
<td>-5 - j5</td>
</tr>
</tbody>
</table>