Chapter 18 KINETICS OF RIGID BODIES IN THREE DIMENSIONS

The two fundamental equations for the motion of a system of particles

\[ \Sigma F = m \ddot{\mathbf{a}} \quad \Sigma M_G = \dot{\mathbf{H}}_G \]

provide the foundation for three dimensional analysis, just as they do in the case of plane motion of rigid bodies. The computation of the angular momentum \( \mathbf{H}_G \) and its derivative \( \dot{\mathbf{H}}_G \), however, are now considerably more involved.

The rectangular components of the angular momentum \( \mathbf{H}_g \) of a rigid body may be expressed in terms of the components of its angular velocity \( \omega \) and of its centroidal moments and products of inertia:

\[ \begin{align*}
H_x &= +I_x \omega_x - I_{xy} \omega_y - I_{xz} \omega_z \\
H_y &= -I_{yx} \omega_x + I_y \omega_y - I_{yz} \omega_z \\
H_z &= -I_{zx} \omega_x - I_{zy} \omega_y + I_z \omega_z 
\end{align*} \]

If principal axes of inertia \( G_{x'y'z'} \) are used, these relations reduce to

\[ \begin{align*}
H_{x'} &= I_{x'} \omega_{x'} \\
H_{y'} &= I_{y'} \omega_{y'} \\
H_{z'} &= I_{z'} \omega_{z'} 
\end{align*} \]
In general, the angular momentum $H_G$ and the angular velocity $\omega$ do not have the same direction. They will, however, have the same direction if $\omega$ is directed along one of the principal axes of inertia of the body.

The system of the momenta of the particles forming a rigid body may be reduced to the vector $m\vec{v}$ attached at $G$ and the couple $H_G$. Once these are determined, the angular momentum $H_O$ of the body about any given point $O$ may be obtained by writing

$$H_O = \vec{r} \times m\vec{v} + H_G$$

In the particular case of a rigid body constrained to rotate about a fixed point $O$, the components of the angular momentum $H_O$ of the body about $O$ may be obtained directly from the components of its angular velocity and from its moments and products of inertia with respect to axes through $O$.

$$H_x = +I_x \omega_x - I_{xy} \omega_y - I_{xz} \omega_z$$
$$H_y = -I_{yx} \omega_x + I_y \omega_y - I_{yz} \omega_z$$
$$H_z = -I_{zx} \omega_x - I_{zy} \omega_y + I_z \omega_z$$
The *principle of impulse and momentum* for a rigid body in three-dimensional motion is expressed by the same fundamental formula used for a rigid body in plane motion.

\[
\text{Syst Momenta}_1 + \text{Syst Ext Imp}_1 \rightarrow \text{Syst Momenta}_2
\]

The initial and final system momenta should be represented as shown in the figure and computed from

\[
\begin{align*}
H_x &= \bar{I}_x' \omega_x - I_{xy} \omega_y - I_{xz} \omega_z \\
H_y &= -I_{yx} \omega_x + \bar{I}_y' \omega_y - I_{yz} \omega_z \\
H_z &= -I_{zx} \omega_x - I_{zy} \omega_y + \bar{I}_z' \omega_z
\end{align*}
\]

or

\[
\begin{align*}
H_x' &= \bar{I}_x' \omega_x, \\
H_y' &= \bar{I}_y' \omega_y, \\
H_z' &= \bar{I}_z' \omega_z
\end{align*}
\]

The kinetic energy of a rigid body in three-dimensional motion may be divided into two parts, one associated with the motion of its mass center \(G\), and the other with its motion about \(G\). Using principal axes \(x', y', z'\), we write

\[
T = \frac{1}{2} m \overline{v}^2 + \frac{1}{2} \left( \bar{I}_x' \omega_x^2 + \bar{I}_y' \omega_y^2 + \bar{I}_z' \omega_z^2 \right)
\]

where

- \(\overline{v}\) = velocity of the mass center
- \(\omega\) = angular velocity
- \(m\) = mass of rigid body
- \(\bar{I}_x', \bar{I}_y', \bar{I}_z'\) = principal centroidal moments of inertia.
In the case of a rigid body *constrained to rotate about a fixed point* $O$, the kinetic energy may be expressed as

$$T = \frac{1}{2} \left( I_x \omega_x^2 + I_y \omega_y^2 + I_z \omega_z^2 \right)$$

where $x'$, $y'$, and $z'$ axes are the principal axes of inertia of the body at $O$. The equations for kinetic energy make it possible to extend to the three-dimensional motion of a rigid body the application of the *principle of work and energy* and of the *principle of conservation of energy*.

The fundamental equations

$$\Sigma F = m\ddot{a} \quad \Sigma M_G = \dot{H}_G$$

can be applied to the motion of a rigid body in three dimensions. We first recall that $H_G$ represents the angular momentum of the body relative to a centroidal frame $GX'Y'Z'$ of fixed orientation and that $\dot{H}_G$ represents the rate of change of $H_G$ with respect to that frame. As the body rotates, its moments and products of inertia with respect to $GX'Y'Z'$ change continually. It is therefore more convenient to use a frame $Gxyz$ rotating with the body to resolve $\omega$ into components and to compute the moments and products of inertia which are used to determine $H_G$. 
\[ \Sigma F = m\ddot{a} \quad \Sigma M_G = \dot{H}_G \]

\( \dot{H}_G \) represents the rate of change of \( H_G \) with respect to the frame \( GX'Y'Z' \) of fixed orientation, therefore

\[ \dot{H}_G = (\dot{H}_G)_{Gxyz} \times \Omega \]

where \( H_G \) = angular momentum of the body with respect to the frame \( GX'Y'Z' \) of fixed orientation
\( (\dot{H}_G)_{Gxyz} \) = rate of change of \( H_G \) with respect to the rotating frame \( Gxyz \)
\( \Omega \) = angular velocity of the rotating frame \( Gxyz \)

If the rotating frame is attached to the body, its angular velocity \( \Omega \) is identical to the angular velocity \( \omega \) of the body.

Setting \( \Omega = \omega \), using principal axes, and writing this equation in scalar form, we obtain Euler's equations of motion.
In the case of a rigid body constrained to rotate about a fixed point \( O \), an alternative method of solution may be used, involving moments of the forces and the rate of change of the angular momentum about point \( O \).

\[
\Sigma M_O = (\dot{H}_O)_{Oxyz} + \Omega \times H_O
\]

where \( \Sigma M_O \) = sum of the moments about \( O \) of the forces applied to the rigid body

- \( H_O \) = angular momentum of the body with respect to the frame \( OXYZ \)
- \( (\dot{H}_O)_{Oxyz} \) = rate of change of \( H_O \) with respect to the rotating frame \( Oxyz \)
- \( \Omega \) = angular velocity of the rotating frame \( Oxyz \).

When the motion of gyroscopes and other axisymmetrical bodies are considered, the Eulerian angles \( \phi, \theta, \) and \( \psi \) are introduced to define the position of a gyroscope. The time derivatives of these angles represent, respectively, the rates of precession, nutation, and spin of the gyroscope. The angular velocity \( \omega \) is expressed in terms of these derivatives as

\[
\omega = -\dot{\phi} \sin \theta \, i + \dot{\theta} \, j + (\dot{\psi} + \dot{\phi} \cos \theta) \, k
\]
\[ \omega = -\dot{\phi} \sin \theta \mathbf{i} + \dot{\theta} \mathbf{j} + (\dot{\psi} + \dot{\phi} \cos \theta) \mathbf{k} \]

The unit vectors are associated with the frame \( O_{xyz} \) attached to the inner gimbal of the gyroscope (figure to the right) and rotate, therefore, with the angular velocity

\[ \Omega = -\dot{\phi} \sin \theta \mathbf{i} + \dot{\theta} \mathbf{j} + \dot{\phi} \cos \theta \mathbf{k} \]

Denoting by \( I \) the moment of inertia of the gyroscope with respect to its spin axis \( z \) and by \( I' \) its moment of inertia with respect to a transverse axis through \( O \), we write

\[ \mathbf{H}_O = -I'\dot{\phi} \sin \theta \mathbf{i} + I\dot{\theta} \mathbf{j} + I(\dot{\psi} + \dot{\phi} \cos \theta) \mathbf{k} \]

Substituting for \( \mathbf{H}_O \) and \( \Omega = -\dot{\phi} \sin \theta \mathbf{i} + \dot{\theta} \mathbf{j} + \dot{\phi} \cos \theta \mathbf{k} \) into

\[ \Sigma \mathbf{M}_O = (\dot{\mathbf{H}}_O)_{Oxyz} + \mathbf{\Omega} \times \mathbf{H}_O \]

leads to the differential equations defining the motion of the gyroscope.
In the particular case of the *steady precession* of a gyroscope, the angle $\theta$, the rate of precession $\dot{\phi}$, and the rate of spin $\psi$ remain constant. Such motion is possible only if the moments of the external forces about $O$ satisfy the relation

$$\Sigma M_O = (I\omega_z - I'\dot{\phi}\cos \theta)\dot{\phi}\sin \theta \mathbf{j}$$

i.e., if the external forces reduce to a couple of moment equal to the right-hand member of the equation above and applied *about an axis perpendicular to the precession axis and to the spin axis.*