**Integral Formulation for Numerical Solutions**

**OBJECTIVE:** Use integral methods to obtain an approximate solution to a physical problem.

**EXAMPLE:** Find a solution for a beam deflection.

\[
H m_0 M(x) = m_0
\]

**Differential Equations**

\[
EI \frac{d^2 y}{dx^2} - M(x) = 0
\]

- **EI** = beam stiffness
- **M(x)** = bending moment
Approximate Beam deflection:

\[ y(x) = \sin \frac{\pi x}{H} \]

Exact Solution:

\[
\frac{d^2 y}{dx^2} = \frac{M(x)}{EI}
\]

Integrating twice

\[
\frac{dy}{dx} = \frac{Mx}{EI} + c
\]

\[ y = \frac{Mx^2}{2EI} + c_1 x + c_2 \]

Using boundary conditions:

\[ y(0) = y(H) = 0 \]

\[ y(0) = 0, x = 0 \quad \iff \quad c_2 = 0 \]

\[ 0 = \frac{MH^2}{2EI} + c_1 H \]

\[ c_1 = -\frac{MH^2}{2EI} = \frac{MH}{2EI} \]

\[ y = \frac{Mx^2}{2EI} - \frac{MHx}{2EI} \]

\[ y = \frac{MHx}{2EI} (x - H) \quad \text{Exact Solution.} \]
VARIATIONAL METHOD

Given a differential equation such as:

\[ D \frac{d^2 y}{dx^2} - Q = 0 \]

The calculus of variations will generate a solution if the numerical value of the integral \( \Pi \) is a minimum.

In other words, if a particular \( y = g(x) \) generates a minimum for \( \Pi \); it is the solution of the differential equation.
So, for the beam:

\[ \Pi = \int_{0}^{H} \left( \frac{d^2y}{dx^2} - M_0y \right) dx \]

Consider the approximate function, \( y(x) = A \sin \frac{\pi x}{H} \)
then evaluate \( \Pi \).

**Steps:**

1. Write \( \Pi \) as a function of \( A \)
2. Minimize with respect to \( A \)

\[ y(x) = A \sin \frac{\pi x}{H} \]
\[ \frac{dy}{dx} = A \frac{\pi}{H} \cos \frac{\pi x}{H} \]
VARIATIONAL METHOD
continued

\[ \Pi = \int_{0}^{H} \left( \frac{E I}{2} \frac{A \pi}{H} \cos \frac{\pi x}{H} - M_{0} A \sin \frac{\pi x}{H} \right) dx \]

\[ \Pi = \frac{E I \pi^{2}}{4 H} \frac{A^{2}}{2} + \frac{2 M_{0} H}{\pi} A \]
Minimizing $\Pi$:

$$\frac{\partial \Pi}{\partial A} = 2EI\frac{\pi^2}{4H}A + \frac{2M_0H}{\pi} = 0$$

$$A = -\frac{4M_0H^2}{\pi^3EI}$$

$$y(x) = -\frac{4M_0H^2}{\pi^3EI}\sin\left(\frac{\pi x}{H}\right)$$
Development of a Continuous Solution
in a 1 dimensional region

Apply shape functions concept to each element

\[ \phi_1 = N_i^{(1)} \Phi_i + N_j^{(1)} \Phi_j = N_1^{(1)} \Phi_1 + N_2^{(1)} \Phi_2 \]
\[ \phi_2 = N_i^{(2)} \Phi_i + N_j^{(2)} \Phi_j = N_2^{(2)} \Phi_2 + N_3^{(2)} \Phi_3 \]
\[ \phi_3 = N_i^{(3)} \Phi_i + N_j^{(3)} \Phi_j = N_3^{(3)} \Phi_3 + N_4^{(3)} \Phi_4 \]
\[ \phi_4 = N_i^{(4)} \Phi_i + N_j^{(4)} \Phi_j = N_4^{(4)} \Phi_4 + N_5^{(4)} \Phi_5 \]
\[ \phi^{(e)}_{le} = N_i^{(e)} \Phi_i + N_j^{(e)} \Phi_j \]

**General Form**

With Shape functions:

\[ N_i^{(e)} = \frac{X_j - x}{X_j - X_i} \quad \& \quad N_j^{(e)} = \frac{x - X_i}{X_j - X_i} \]
Development of a Continuous Solution
in a 1 dimensional region
(continued)

Grid Summary:

<table>
<thead>
<tr>
<th>element</th>
<th>nodes</th>
</tr>
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<tbody>
<tr>
<td>#</td>
<td>i</td>
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<tr>
<td>(1)</td>
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<td>(3)</td>
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<td>(4)</td>
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</tbody>
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Note:

\[ N_2^{(1)} \neq N_2^{(2)} \]
USE OF GALERKIN’S METHOD FOR THE SOLUTION OF ONE DIMENSIONAL DIFFERENTIAL EQUATIONS

**Problem:** Solve equations of the form:

\[ D \frac{d^2 y}{dx^2} - Q = 0 \]

- Solid mechanics problems
- Solution of Differential Equations
  - Beam Deflection
  - Heat flow through a wall

With boundary conditions:

\[ \phi(a) = \phi_o \quad \& \quad \phi(H) = \phi_H \]
Procedure: Evaluate Residual Integral.

\[ \int_0^H W_i(x)R(x)dx = 0 \]

- Obtain a nodal equation for each node
- Apply integral evaluation to each node
- Generate a system of linear equations
- Find beam deflection
USE OF GALERKIN’S METHOD FOR THE SOLUTION OF ONE DIMENSIONAL DIFFERENTIAL EQUATIONS

DEFINE WEIGHTING FUNCTIONS:

- Evaluate the Weighted Residual Integral

\[ - \int_{0}^{H} W(x) \left( E \frac{d^2 \phi}{dx^2} + Q \right) dx = 0 \]

\[ \boxed{R(x)} \quad \text{because } \phi \text{ is not an exact solution} \]
- Use a new weighting function for each node where $\phi$ is unknown.
- Construct weighting functions using the shape functions $N_i$ & $N_j$.
- Weighting functions of the $s^{th}$ node = $W_s$, or $W_s =$ shape functions associated with the $s^{th}$ node.

$$W_3(x) = \begin{cases} N_3^{(2)} & x_2 \leq x \leq x_3 \\ N_3^{(3)} & x_3 \leq x \leq x_4 \end{cases}$$

$$W_s(x) = \begin{cases} N_s^{(e)} & x_r \leq x \leq x_s \\ N_s^{(e+1)} & x_s \leq x \leq x_t \end{cases}$$
EVALUATION OF THE RESIDUAL INTEGRAL:

We need to evaluate

\[- \frac{H}{4} \int_{0}^{L} W(x) E \frac{d^2 \phi}{dx^2} + Q \int_{0}^{L} dx = 0\]

\[R_s = R_s^{(e)} + R_s^{(e+1)}\]

Residual due to element (e)  Residual due to element (e+1)

\[= - \int_{x_r}^{x_s} E \frac{d^2 \phi}{dx^2} + Q \int_{x_r}^{x_t} dx - \int_{x_s}^{x_t} E \frac{d^2 \phi}{dx^2} + Q \int_{x_s}^{x_t} dx = 0\]

Since \(\phi\) is not continuous

\[\therefore \int \frac{d^2 \phi}{dx^2} dx \text{ not defined}\]
But it can be changed to a new term such as:

\[
\frac{d}{dx} \left[ N_s \frac{d\phi}{dx} \right] = N_s \frac{d^2\phi}{dx^2} + \frac{dN_s}{dx} \frac{d\phi}{dx}
\]

Then:

\[
N_s \frac{d^2\phi}{dx^2} = \frac{d}{dx} \left[ N_s \frac{d\phi}{dx} \right] - \frac{dN_s}{dx} \frac{d\phi}{dx}
\]

Now substitute into the integral:
EVALUATION OF THE RESIDUAL INTEGRAL:

(continued)

\[
- \int_{x_r}^{x_s} N D \frac{d^2 \phi}{dx^2} K \, dx = - \int_{x_r}^{x_s} N_s \frac{d \phi}{dx} (e) \, dx + \int_{x_r}^{x_s} dN_s \frac{d \phi}{dx} (e) \, dx
\]

\[
- \int_{x_s}^{x_t} N D \frac{d^2 \phi}{dx^2} K \, dx = - \int_{x_s}^{x_t} N_s \frac{d \phi}{dx} (e+1) \, dx + \int_{x_s}^{x_t} dN_s \frac{d \phi}{dx} (e+1) \, dx
\]

Now we need to evaluate the complete residual.
Complete residual equation: (Interior node)

\[
R_s = R_s^{(e)} + R_s^{(e+1)} = - \int \mathbf{H} \left( x \right) E \frac{d^2 \phi}{dx^2} + Q \, dx
\]
\[ R_s = R_s^{(e)} + R_s^{(e+1)} = -H \int_0^1 W(x) \frac{d^2 \phi}{dx^2} + Q \] 

Evaluation of these establishes an interelement requirement. Difference must be zero so that \( R_s = 0 \).

Remember
\[ N_i = 1 \quad N_j = 0 \]
\[ N_s = 1 \quad N_r = 0 \]

and
\[ N_s = 1 \quad N_t = 0 \]
EVALUATION OF THE INTEGRAL:

(continued)

Remember

\[ \phi^{(e)} = N_r \Phi_r + N_s \Phi_s \]

Shape Functions

\[ \phi^{(e)} = \frac{x - x'}{L} \Phi_r + \frac{X - x'}{L} \Phi_s \]

Thus:

\[ N_s^{(e)} = \frac{x - X_r}{L} \Rightarrow \frac{dN_s^{(e)}}{dx} = \frac{1}{L} \]
This will give the rate of change of $\Phi$ within the element.

So, we want $R_s^{(e)} \& R_s^{(e+1)}$

$$R_s^{(e)} = - \left[ \frac{d\Phi^{(e)}}{dx} \right]_{x=x_s}^{x=x_r} + \int_{x_r}^{x_w} dN_s d\phi \int k dx - N_s Q^{(e)} dx$$

$$= - \left[ \frac{d\Phi^{(e)}}{dx} \right]_{x=x_s}^{x=x_r} + \int_{x_r}^{x_w} dN_s d\phi \int k dx - N_s Q^{(e)} dx$$
EVALUATION OF THE INTEGRAL:
(continued)

\[ R_s^{(e)} = - \left[ 1 \cdot \frac{d\phi}{dx} \right]_{x=x_s}^{(e)} + \left[ \frac{dN_s}{dx} \frac{d\phi}{dx} \right]_{x=x_r}^{(e)} dx - \int_{x_r}^{x_e} N_s Q dx \]

\[ + \int_{x_r}^{x_e} \Phi_r + \Phi_s \Phi dx - \int_{x_r}^{x_e} Q \frac{x-X_r}{L} dx \]

\[ + \int_{x_r}^{x_e} \Phi_r + \Phi_s g \frac{Q}{L} \left[ \frac{X_s^2}{2} - X_r \cdot x \right]_{x=X_s}^{x=X_r} \]

\[ R_s^{(e)} = - \left[ 1 \cdot \frac{d\phi}{dx} \right]_{x=x_s}^{(e)} + \frac{D}{L} b \Phi_r + \Phi_s g \frac{QL}{2} \]
Similarly, we obtain

\[ R_s^{(e+1)} \]

\[ \phi^{(e+1)} = N_s \Phi_s + N_t \Phi_t \]

\[ \phi^{(e+1)} = \frac{X_t - x}{L} \Phi_s + \frac{x - X_t}{L} \Phi_t \]

\[ N_s^{(e+1)} = \frac{X_t - x}{L} \implies \frac{dN_s^{(e+1)}}{dx} = -\frac{1}{L} \]

\[ \frac{d\phi^{(e+1)}}{dx} = \frac{1}{L} (-\Phi_s + \Phi_t) \]
\[ R_s^{(e+1)} = -\left[ \frac{d\phi}{dx} \right]_{x=x_s}^{(e+1)} + \frac{dN_s}{dx} d\phi - N_s Q \int dx \]

\[ + \frac{Q}{L} \int^{x_s} dx \Phi_s + \Phi_t \int^{x_s} dx - Q \frac{x_s - x}{L} \]

\[ + \frac{Q}{L} \Phi_s - \Phi_t g \frac{QL}{2} \]
EVALUATION OF THE INTEGRAL:
(continued)

Finally the residual equation for node “s”:

\[ R_s = -\left. \frac{d\phi}{dx} \right|_{x=x_s}^{(e+1)} + \left. \frac{d\phi}{dx} \right|_{x=x_t}^{(e)} k - \text{Interelement requirement} \]

\[ -\frac{D^{(e)}}{L} \Phi_r + \frac{D^{(e)}}{L} + \frac{D^{(e+1)}}{L} \Phi_t - \frac{D^{(e+1)}}{L} \]

\[ -\frac{\Omega L^{(e)}}{k} - \frac{\Omega L^{(e+1)}}{k} = 0 \]
Observation about the interelement requirement.
- Can be used to evaluate quality of grid.
- Indicate where grid should be refined.
- If $D^{(e)} = D^{(e+1)}$ \[ \Rightarrow \left. \frac{d\phi}{dx} \right|_{x=X_s}^{(e+1)} - \left. \frac{d\phi}{dx} \right|_{x=X_s}^{(e)} \]
- It can be viewed as an error term.
- Not incorporated into system of equations.
- Remind us that solution is approximate.

So finally the nodal residual equation is:

\[ R_s = - \frac{Q_{L}^{(s-1)}}{k} \Phi_{s-1} + \frac{Q_{L}^{(s-1)}}{L} \frac{D^{(s)}}{L} \Phi_{s} - \frac{D^{(s)}}{L} \Phi_{s+1} \]

\[ - \frac{Q_{L}^{(s-1)}}{k} - \frac{Q_{L}^{(s)}}{k} = 0 \]

**Nodal Residual equation for node “s”**