2.5.2 \( \lambda^2 + 6\lambda + 25 = (\lambda + 3 - j4)(\lambda + 3 + j4) \) characteristic roots are \(-3 \pm j4\)

\[ y_0(t) = Ke^{-3t} \cos(4t + \theta) \]

For \( f(t) = u(t) \), \( y_0(t) = H(0) = \frac{3}{2} \) so that

\[ y(t) = Ke^{-3t} \cos(4t + \theta) + \frac{3}{25} \]
\[ \dot{y}(t) = -3Ke^{-3t} \cos(4t + \theta) - 4Ke^{-3t} \cos(4t + \theta) \]

Setting \( t = 0 \), and substituting initial conditions yields

\[ \left. \begin{array}{l}
0 = K \cos \theta + \frac{3}{25} \\
2 = 3K \cos \theta - 4K \sin \theta
\end{array} \right\} \implies \begin{cases} 
K \cos \theta = \frac{3}{25} \\
K \sin \theta = \frac{19}{25}
\end{cases} \]
\[ K = 0.427 \]
\[ \theta = -106.3 \]

and

\[ y(t) = 0.427e^{-3t} \cos(4t - 106.3^\circ) + \frac{3}{25} \quad t \geq 0 \]

2.5.4 Because \( (\lambda^2 + 2\lambda) = \lambda(\lambda + 2) \), the characteristic roots are 0 and -2.

\[ y_0(t) = K_1 + K_2 e^{-2t} \]

In this case \( f(t) = u(t) \). The input itself is a characteristic mode. Therefore

\[ y_0(t) = \beta t \]

But \( y_0(t) \) satisfies the system equation

\[ (D^2 + 2D)y_0(t) = (D + 1)y(t) = \ddot{y}_0(t) + 2\dot{y}_0(t) = f(t) + f(t) \]

Substituting \( f(t) = u(t) \) and \( y_0(t) = \beta t \), we obtain

\[ 0 + 2\beta = 0 + 1 \implies \beta = \frac{1}{2} \]

Therefore \( y_0(t) = \frac{1}{2} t \).

\[ y(t) = K_1 + K_2 e^{-2t} + \frac{1}{2} t \]
\[ \dot{y}(t) = -2K_2 e^{-2t} + \frac{1}{2} \]

Setting \( t = 0 \), and substituting initial conditions yields

\[ \left. \begin{array}{l}
2 = K_1 + K_2 \\
1 = -2K_2 + \frac{1}{2}
\end{array} \right\} \implies \begin{cases} 
K_1 = \frac{3}{4} \\
K_2 = -\frac{1}{4}
\end{cases} \]

and

\[ y(t) = \frac{3}{4} - \frac{1}{4} e^{-2t} + \frac{1}{2} t \quad t \geq 0 \]

2.6.1

(a) \( \lambda^2 + 8\lambda + 12 = (\lambda + 2)(\lambda + 6) \)
Both roots are in LHP. The system is asymptotically stable.

(b) \( \lambda^2 + 3\lambda + 2 = \lambda(\lambda + 1)(\lambda + 2) \)
Roots are 0, -1, -2. One root on imaginary axis and none in RHP. The system is marginally stable.

(c) \( \lambda^2(\lambda^2 + 2) = \lambda^3(\lambda + j\sqrt{2})(\lambda - j\sqrt{2}) \)
Roots are 0 (repeated twice) and \( \pm j\sqrt{2} \). Multiple roots on imaginary axis. The system is unstable.

(d) \( (\lambda + 1)(\lambda^2 - 6\lambda + 5) = (\lambda + 1)(\lambda - 1)(\lambda - 5) \)
Roots are -1, 1 and 5. Two roots in RHP. The system is unstable.

3.1.5 (a) If \( x(t) \) and \( y(t) \) are orthogonal, then we showed [see Eq. (3.22)] the energy of \( x(t) + y(t) \) is \( E_x + E_y \). We now find the energy of \( x(t) - y(t) \):

\[
\int_{-\infty}^{\infty} |x(t) - y(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt + \int_{-\infty}^{\infty} |y(t)|^2 dt - 2 \int_{-\infty}^{\infty} x(t)^* y(t) dt - \int_{-\infty}^{\infty} x^*(t) y(t) dt
\]

\[
= \int_{-\infty}^{\infty} |x(t)|^2 dt + \int_{-\infty}^{\infty} |y(t)|^2 dt \tag{3.22}
\]
The last result follows from the fact that because of orthogonality, the two integrals of the cross products \(x(t)y'(t)\) and \(x'(t)y(t)\) are zero [see Eq. (3.20)]. Thus the energy of \(x(t) + y(t)\) is equal to that of \(x(t) - y(t)\) if \(x(t)\) and \(y(t)\) are orthogonal.

(b) Using similar argument, we can show that the energy of \(c_1x(t) + c_2y(t)\) is equal to that of \(c_1x(t) - c_2y(t)\) if \(x(t)\) and \(y(t)\) are orthogonal. This energy is given by \(|c_1|^2E_x + |c_2|^2E_y\).

(c) If \(z(t) = x(t) \pm y(t)\), then

\[
\int_{-\infty}^{\infty} |x(t) \pm y(t)|^2 \, dt = \int_{-\infty}^{\infty} |x(t)|^2 \, dt + \int_{-\infty}^{\infty} |y(t)|^2 \, dt \pm \int_{-\infty}^{\infty} x(t)y'(t) \, dt \pm \int_{-\infty}^{\infty} x'(t)y(t) \, dt
\]

\[
= E_x \pm E_y \pm (E_{xy} + E_{yx})
\]

3.4-1 Here \(T_0 = 2\), so that \(\omega_0 = 2\pi/2 = \pi\), and

\[
f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi t + b_n \sin n\pi t \quad -1 \leq t \leq 1
\]

where

\[
a_0 = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f(t) \, dt = \frac{1}{2} \int_{-1}^{1} t^2 \cos n\pi t \, dt = \frac{4(-1)^n}{\pi n^2}, \quad b_n = \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} f(t) \sin n\pi t \, dt = 0
\]

Therefore

\[
f(t) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi t \quad -1 \leq t \leq 1
\]

Figure S3.4-1 shows \(f(t) = t^2\) for all \(t\) and the corresponding Fourier series representing \(f(t)\) over \((-1, 1)\).

3.4-3 (a) \(T_0 = 4\), \(\omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{2}\). Because of even symmetry, all sine terms are zero.

\[
f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos \left(\frac{n\pi}{2} t\right)
\]

\[
a_0 = 0 \text{ (by inspection)}
\]

\[
a_n = \frac{2}{4} \left[ \int_{0}^{1} \cos \left(\frac{n\pi}{2} t\right) \, dt - \int_{1}^{2} \cos \left(\frac{n\pi}{2} (t-1)\right) \, dt \right] = \frac{4}{n\pi} \sin \frac{n\pi}{2}
\]

Therefore, the Fourier series for \(f(t)\) is

\[
f(t) = \frac{4}{\pi} \left[ \cos \frac{\pi t}{2} - \frac{1}{3} \cos \frac{3\pi t}{2} + \frac{1}{5} \cos \frac{5\pi t}{2} - \frac{1}{7} \cos \frac{7\pi t}{2} + \ldots \right]
\]

Here \(b_n = 0\), and we allow \(C_n\) to take negative values. Figure S3.4-3a shows the plot of \(C_n\).

(b) \(T_0 = 10\pi\), \(\omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{5}\). Because of even symmetry, all the sine terms are zero.

\[
f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos \left(\frac{n\pi}{5} t\right) + b_n \sin \left(\frac{n\pi}{5} t\right)
\]

\[
a_0 = \frac{1}{5} \text{ (by inspection)}
\]

\[
a_n = \frac{2}{10\pi} \left[ \int_{-\infty}^{\infty} \cos \left(\frac{n\pi}{5} t\right) \, dt - \int_{\infty}^{\infty} \cos \left(\frac{n\pi}{5} (t-\infty)\right) \, dt \right] = \frac{2}{\pi n} \sin \left(\frac{n\pi}{5}\right)
\]

\[
b_n = \frac{2}{10\pi} \int_{-\infty}^{\infty} \sin \left(\frac{n\pi}{5} t\right) \, dt = 0 \quad \text{(integrand is an odd function of } t)\]

Here \(b_n = 0\), and we allow \(C_n\) to take negative values. Note that \(C_n = a_n\) for \(n = 0, 1, 2, 3, \ldots\). Figure S3.4-3b shows the plot of \(C_n\).

(c) \(T_0 = 2\pi\), \(\omega_0 = 1\).
\[ f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt \quad \text{with} \quad a_0 = 0.5 \quad \text{(by inspection)} \]

\[ a_n = \frac{1}{\pi} \int_{0}^{\pi} f(t) \cos nt \, dt = 0, \quad b_n = \frac{1}{\pi} \int_{0}^{2\pi} f(t) \sin nt \, dt = -\frac{1}{\pi n} \]

and

\[ f(t) = 0.5 - \frac{1}{\pi} \left( \sin t + \frac{1}{2} \sin 2t + \frac{1}{3} \sin 3t + \frac{1}{4} \sin 4t + \ldots \right) \]

\[ = 0.5 + \frac{1}{\pi} \left[ \cos \left( t + \frac{\pi}{2} \right) + \frac{1}{2} \cos \left( 2t + \frac{\pi}{2} \right) + \frac{1}{3} \cos \left( 3t + \frac{\pi}{2} \right) + \ldots \right] \]

The reason for the vanishing of the cosine terms is that when 0.5 (the dc component) is subtracted from \( f(t) \), the remaining function has odd symmetry. Hence, the Fourier series would contain only sine terms. Figure S3.4–3c shows the plot of \( C_n \) and \( \theta_n \).

\( \theta_n = \pi, \omega_n = 2 \) and \( f(t) = \frac{1}{2} t \).

\( a_n = 0 \quad \text{(by inspection)} \)

\( \theta_n = 0 \quad (n > 0) \quad \text{because of odd symmetry} \)

\[ b_n = \frac{4}{\pi} \int_{0}^{\pi} \frac{1}{2} t \sin 2nt \, dt = \frac{2}{\pi n} \left( \frac{\pi n}{2} \sin \frac{\pi n}{2} - \cos \frac{\pi n}{2} \right) \]

\[ f(t) = \frac{4}{\pi} \sin 2t + \frac{1}{2} \sin 4t - \frac{4}{\sqrt{3}} \sin 6t + \frac{1}{2} \sin 8t + \ldots \]

\[ = \frac{4}{\pi} \cos \left( 2t - \frac{\pi}{2} \right) + \frac{1}{2} \cos \left( 4t - \frac{\pi}{2} \right) + \frac{4}{\sqrt{3}} \cos \left( 6t + \frac{\pi}{2} \right) + \frac{1}{2} \cos \left( 8t + \frac{\pi}{2} \right) + \ldots \]

Figure S3.4–3d shows the plot of \( C_n \) and \( \theta_n \).
(e) $T_0 = 3$, $\omega_0 = 2\pi/3$.

$$a_0 = \frac{1}{3} \int_0^1 t \, dt = \frac{1}{6}$$

$$a_n = \frac{2}{3} \int_0^1 t \cos \frac{2\pi n}{3} \, dt = \frac{3}{2\pi^2 n^2} \left[ \cos \frac{2\pi n}{3} + \frac{2\pi n}{3} \sin \frac{2\pi n}{3} - 1 \right]$$

$$b_n = \frac{2}{3} \int_0^1 t \sin \frac{2\pi n}{3} \, dt = \frac{3}{2\pi^2 n^2} \left[ \sin \frac{2\pi n}{3} - \frac{2\pi n}{3} \cos \frac{2\pi n}{3} \right]$$

Therefore $C_0 = \frac{1}{6}$ and

$$C_n = \frac{3}{2\pi^2 n^2} \left[ \sqrt{2 - \left( \frac{4\pi^2 n^2}{9} - 2 \cos \frac{2\pi n}{3} \sin \frac{2\pi n}{3} \right)} \right] \quad \text{and} \quad \theta_n = \tan^{-1} \left( \frac{2\pi n}{3} \cos \frac{2\pi n}{3} - \sin \frac{2\pi n}{3} \right)$$

(f) $T_0 = 6$, $\omega_0 = \pi/3$, $a_0 = 0.5$ (by inspection). Even symmetry; $b_n = 0$.

$$u_n = \frac{4}{9} \int_0^1 f(t) \cos \frac{n\pi t}{3} \, dt$$

$$= \frac{2}{3} \left[ \int_0^1 \cos \frac{n\pi t}{3} \, dt + \int_1^2 (2-t) \cos \frac{n\pi t}{3} \, dt \right]$$

$$= \frac{6}{\sqrt{5} \pi n^2} \left[ \cos \frac{n\pi}{3} - \cos \frac{2n\pi}{3} \right]$$

$$f(t) = 0.5 + \frac{6}{\sqrt{5} \pi} \left( \cos \frac{\pi t}{3} - \frac{2}{9} \cos \pi t + \frac{1}{21} \cos \frac{5\pi t}{4} - \frac{1}{29} \cos \frac{7\pi t}{3} + \cdots \right)$$

Observe that even harmonics vanish. The reason is that if the dc (0.5) is subtracted from $f(t)$, the resulting function has half-wave symmetry. (See Prob. 3.4-7). Figure 3.4-3f shows the plot of $C_n$.

3.4-5 (a) Here $T_0 = \pi/2$, and $\omega_0 = \frac{2\pi}{T_0} = 4$. Therefore

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos 4nt + b_n \sin 4nt$$

where

$$a_0 = \frac{2}{\pi} \int_0^{\pi/2} e^{-t} \, dt = 0.504$$

$$a_n = \frac{4}{\pi} \int_0^{\pi/2} e^{-t} \cos 4nt \, dt = 0.504 \left( \frac{2}{1 + 16n^2} \right)$$

and

$$b_n = \frac{4}{\pi} \int_0^{\pi/2} e^{-t} \sin 4nt \, dt = 0.504 \left( \frac{8n}{1 + 16n^2} \right)$$

Therefore
\[ C_0 = c_o = 0.504, \quad C_n = \sqrt{\frac{2}{1 + \frac{1}{10n^2}}} \quad \theta_n = -\tan^{-1} \frac{1}{n} \]

(b) This Fourier series is identical to that in Example 3.5(a) with \( t \) replaced by \( \Omega t \).

(c) If \( f(t) = C_0 + \sum C_n \cos(n\omega_0 t + \theta_n) \), then

\[ f(at) = C_0 + \sum C_n \cos(na\omega_0 t + \theta_n) \]

Thus, time scaling by a factor \( a \) merely scales the fundamental frequency by the same factor \( a \). Everything else remains unchanged. If we time compress (or time expand) a periodic signal by a factor \( a \), its fundamental frequency increases by the same factor \( a \) (or decreases by the same factor \( a \)). Comparison of the results in part (a) with those in Example 3.3 confirms this conclusion. This result applies equally well.

3.5-1 (a): \( T_0 = 4, \omega_0 = \pi/2 \). Also \( D_0 = 0 \) (by inspection).

\[ D_n = \frac{1}{2\pi} \int_{-1}^{1} e^{-j(n\pi/2)t} dt - \int_{1}^{3} e^{-j(n\pi/2)t} dt = \frac{2}{\pi n} \sin \frac{n\pi}{2} \quad |n| \geq 1 \]

(b) \( T_0 = 10\pi, \omega_0 = 2\pi/10\pi = 1/5 \)

\[ f(t) = \sum_{n=-\infty}^{\infty} D_n e^{j2\pi t}, \quad \text{where} \quad D_n = \frac{1}{10\pi} \int_{0}^{10\pi} e^{-j\frac{n\pi t}{10}} dt = \frac{i}{2\pi} \left( -2j \sin \frac{n\pi}{5} \right) = \frac{1}{\pi n} \sin \left( \frac{n\pi}{5} \right) \]

(c)

\[ f(t) = D_0 + \sum_{n=-\infty}^{\infty} D_n e^{j2\pi t}, \quad \text{where, by inspection} \quad D_0 = 0.5 \]

\[ D_n = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-j\pi n t} dt = \frac{i}{2\pi n}, \quad \text{so that} \quad |D_n| = \frac{1}{2\pi n}, \quad \text{and} \quad \theta D_n = \begin{cases} \frac{\pi}{2} & n > 0 \\ -\frac{\pi}{2} & n < 0 \end{cases} \]

(d) \( T_0 = \pi, \omega_0 = 2 \) and \( D_n = 0 \)

\[ f(t) = \sum_{n=-\infty}^{\infty} D_n e^{j2\pi n t}, \quad \text{where} \quad D_n = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{-j2\pi nt} dt = \frac{-j}{2\pi n} \left( \frac{2}{\pi n} \sin \frac{n\pi}{2} - \cos \frac{n\pi}{2} \right) \]

(e) \( T_0 = 3, \omega_0 = \frac{2\pi}{3} \)

\[ f(t) = \sum_{n=-\infty}^{\infty} D_n e^{j\frac{2\pi n}{3} t}, \quad \text{where} \quad D_n = \frac{1}{3} \int_{0}^{1} e^{-j\frac{2\pi n}{3} t} dt = \frac{3}{3\pi n^2} \left[ e^{-j\frac{2\pi n}{3}} \left( \frac{j2\pi n}{3} + 1 \right) - 1 \right] \]
Therefore

\[ |D_n| = \frac{3}{4\pi^2 n^2} \left[ \sqrt{\frac{4\pi^2 n^2}{9}} - 2 \cos \frac{2\pi n}{3} - \frac{4\pi n}{3} \sin \frac{2\pi n}{3} \right] \]

and

\[ \angle D_n = \tan^{-1} \left( \frac{2\pi n \cos \frac{2\pi n}{3} - \sin \frac{4\pi n}{3}}{\cos \frac{2\pi n}{3} + \frac{4\pi n}{3} \sin \frac{4\pi n}{3} - 1} \right) \]

(f) \( T_0 = 0, \omega_0 = \pi/3, D_0 = 0.5 \)

\[ f(t) = 0.5 + \sum_{n=-\infty}^{\infty} D_n e^{i\omega nt} \]

\[ D_n = \frac{1}{6} \left[ \int_{-2}^{-1} (t+2)e^{-it\pi/3} dt + \int_{-1}^{0} e^{-it\pi/3} dt + \int_{0}^{1} (-t+2)e^{-it\pi/3} dt \right] = \frac{3}{4\pi^2} \left( \cos \frac{\pi n}{3} - \cos \frac{2\pi n}{3} \right) \]