MORPHOLOGICAL OPERATIONS

A real image has continuous intensity. It is quantized to obtain a digital image with a given number of gray levels. Different numbers such as 256 (most common) and 512 are used. Higher numbers such as 4096 are used in more sensitive applications (medical imaging for example). In the early days of machine vision, storage memory and computer power were very limited and expensive. For this reason, design focused on binary images (two levels of gray). Although computer memory and speed have improved considerably, binary images are still widely used in various applications.

- Applications of binary images:
  - Chromosome analysis
  - Optical character recognition
  - Industrial parts recognition

- Advantages of binary images
  - Smaller memory requirements
  - Faster execution time
  - Can be used in many applications

- Limitations
  - Cannot be extended to 3D
  - Losing intensity details (information).

A. Thresholding operations

Thresholding allows to obtain a binary image from a gray level image. Thresholding can be expressed as follows

\[ A(i, j) = \begin{cases} 1 & \text{if } f(i, j) < T \\ 0 & \text{otherwise} \end{cases} \] (1)

or

\[ A(i, j) = \begin{cases} 1 & \text{if } f(i, j) \in Z \\ 0 & \text{otherwise} \end{cases} \] (2)

where \( Z \) is a set of intensity levels for a given object. Matlab command \texttt{im2bw} allows to obtain a binary image from a gray level image. Morphological operations are operations on binary images. They affect the shape or structure of the object.

Definition: Morphological operations: analysis and processing of geometric structure.

Mathematical morphology is mainly applied to digital images but it has other applications as well. Mathematical morphology was initially developed for binary images, later, it was extended to gray level images. It was introduced in 1964 by Martheron and Serra, from Ecole des Mines de Paris.

B. Basic concepts from set theory

1) \( a \) is an element of \( A \): \( a \in A \)
2) \( a \) is not an element of \( A \): \( a \notin A \)
3) \( A \) is a subset of \( B \): \( A \subseteq B \)
4) \( C \) is the union of \( A \) and \( B \): \( C = A \cup B \)
5) Union: \( A \cup B = \{p/p \in A \text{ or } p \in B\} \)
6) \( C \) is the intersection of \( A \) and \( B \): \( C = A \cap B \)
7) Intersection: \( A \cap B = \{p/p \in A \text{ and } p \in B\} \)
8) Complement: \( \Omega = \{p/p \in \Omega \text{ and } p \notin A\} \). In a binary image this corresponds to interchanging 0s and 1s.

There are two basic morphological operations: dilation and erosion. Dilation and erosion are performed by laying the structuring element \( B \) on the image \( A \), and sliding it cross the image.

Notations

- Black pixel: value 0.
- White pixel: value 1.
- The structuring element \( B \) is similar to a mask or a filter. \( B \) can have any shape, typical shapes include: row(3), column(3), cross(3).

Matlab function \texttt{strel} can be used to create the structuring element. The size and shape of the structuring element are important factors that affect the output image and cannot be neglected or ignored.

C. Dilation

Dilation can be defined as follows

\[ A \oplus B = \{c/c = a + b, a \in A, b \in B\} \] (3)

where \( A \) is the image and \( B \) is the structuring element. This operation has some similarities with the Minkowski sum.

D. Dilation example

\[ A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \] (4)

and

\[ B = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \] (5)

The result of dilation is

\[ C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \] (6)

Note the the result of convolution will be

\[ C = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \] (7)

Thus, in morphological operations \( 1+1=1 \).

E. Erosion

Erosion is the dual of dilation. It is given by

\[ C = A \ominus B = \{C/B_c \subseteq A\} \] (8)
where $B_c$ is a shift by $c$ units. The structuring element $B$ is positioned with its origin at pixel $(i, j)$ and the new pixel value is determined using the rule

$$C(i, j) = \begin{cases} 1 & \text{if } B \text{ fits in } A \\ 0 & \text{otherwise} \end{cases}$$ (9)

Another way to write erosion is as follows

$$C = A \ominus B = \{x : B_x \in A\}$$ (10)

Therefore, $C$ consists of all points $x$ for which the translation of $B$ by $x$ fits inside $A$.

**F. Some properties of erosion and dilation**

Dilation: is commutative

$$A \oplus B = B \oplus A$$ (11)

Dilation is associative

$$A \oplus (B \oplus C) = (A \oplus B) \oplus C$$ (12)

Erosion is not commutative

$$A \ominus B \neq B \ominus A$$ (13)

**G. Example of erosion and dilation**

Image $A$ is given by

$$A = [1 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 0 \ 1 \ 1]$$ (14)

With

$$B = [1 \ 1 \ 1]$$ (15)

- Erosion:
  $$C = [0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0]$$ (16)

- Dilation:
  $$C = [1 \ 1 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1]$$ (17)

**H. Rules for padding images**

Processing pixels at image border is done as follows:

1) **Dilation**: Pixels beyond the image border are assigned minimum value, that is zero.

2) **Erosion**: Pixels beyond the image border are assigned maximum value, that is 1 or 255 (in gray level).

The structuring element can be of any size or shape, but usually it is much smaller than the image. The structuring element is binary and has a well defined origin (center). In general $B$ is a 3 by 3 mask and it is applied to every pixel in the image.

**I. Examples of structuring element**

The most common structuring elements can be divided into four groups:

1) **SE of Type I**

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$ (18)

2) **SE of Type II**:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$ (19)

3) **SE of Type III, $B_k, k = 1, 2, \ldots, 4$, (rotate $90^\circ$)**:

$$\begin{bmatrix} 1 & X & 1 \\ 1 & 0 & X \\ 1 & X & X \end{bmatrix}$$ (20)

4) **SE of Type IV, $B_k, k = 1, 2, \ldots, 8$, (rotate $45^\circ$)**:

$$\begin{bmatrix} 0 & 0 & 0 \\ X & 1 & X \\ 1 & 1 & 1 \end{bmatrix}$$ (21)

where $X$ means ‘don’t care’

**J. Effects of dilation and erosion**

Dilation expands the white and erosion shrinks the white.

1) Dilation can be used to
   - expand object
   - fill in holes
   - connect disjoint objects

2) Erosion can be used to
   - shrink object

Dilation and erosion are the most basic morphological operations. Assuming the background is black (0) and the object is white (1), dilation expands the object, fills in holes, repairs breaks; erosion shrinks objects, splits apart joined objects, and strips away extrusions. Erosion and dilation are illustrated in figure 1. Both dilation and erosion use a structuring element. Other morphological operations can be derived based on combinations of dilation and erosion.

**K. Example**

For image $A$ in figure 3, apply the following:
Other definitions for erosion and dilation:
- Dilation:
  \[ A \oplus B = \bigcup_{b \in B} A_b \] (22)
  where \( A_b \) is a translation of \( A \) by \( b \) units.
- Erosion:
  \[ A \ominus B = \bigcap_{b \in B} A_{-b} \] (23)
  where \( A_{-b} \) is a translation of \( A \) by \(-b\) units. \( A_b \) is the reverse translation of \( A_{-b} \).

Relationship between dilation and erosion can be written as
\[ A \oplus B = (A^c \ominus B^*)^c \] (24)
where \( B^* \) is the symmetric of \( B \).

L. Example

In order to illustrate equation (24), we consider image \( A \) given by:
\[ A = \begin{bmatrix}
0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 0
\end{bmatrix} \] (25)
and
\[ B = \begin{bmatrix}
1 & 0 & 1 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \] (26)

In this case, the compliment of \( A \) is
\[ A^c = \begin{bmatrix}
1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1
\end{bmatrix} \] (27)
The symmetric of the structuring element is given by
\[ B^s = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \] (28)
and
\[ A^c \odot B^s = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \] (29)
and finally
\[ (A^c \odot B^s)^c = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \] (30)
which is the same as \( A \oplus B \).

Dilation and erosion are the most basic morphological operations. Other operations can be derived based on these operations.

### M. Opening and closing

These are morphological operations derived from dilation and erosion.

- Opening: Erosion followed by dilation using the same structuring element:
  \[ \text{open}(A, B) = \text{dilate}[\text{erode}(A, B), B] \] (31)
or
  \[ A \circ B = (A \odot B) \oplus B \] (32)
- Closing: Dilation followed by erosion using the same structuring element:
  \[ \text{close}(A, B) = \text{erode}[\text{dilate}(A, B), B] \] (33)
or
  \[ A \bullet B = (A \oplus B) \odot B \] (34)
- Both closing and opening use erosion and dilation but in different orders. The order is important, it is what makes the difference.
- Opening and closing are not opposite of each other, they are duals. They are related as follows
  \[ A \bullet B = (A^c \circ B^s)^c \] (35)

where
- \( X^s \) is the symmetric of \( X \)
- \( X^c \) is the complement of \( X \)
- Closing: fills in holes and small gaps, it can be used to remove noise.
- Opening: Remove small objects, can be used for template matching.
- Both closing and opening are idempotent, i.e., applying them multiple times does not change the result.
  \[ (A \circ B) \circ B = A \circ B \] (36)
  \[ (A \bullet B) \bullet B = A \bullet B \] (37)
- Opening and closing satisfy the following
  - Opening: \( A \circ B \subseteq A \)
  - Closing: \( A \subseteq A \bullet B \)

### N. Why opening and closing?

- Erosion is used to remove little objects in the image but at the same time, after erosion the object is smaller. Applying dilation with the same structuring element leads to the original size of the objects.
- Dilation fills little holes, but it also enlarges the object. An erosion brings the object to the original size.

### O. Morphological gradient

Morphological gradient acts in a similar way to the gradient, i.e., can be used to detect edges, i.e., boundaries in binary images.

\[ \nabla A = \frac{(A \oplus B) - (A \odot B)}{\text{diam } B} \] (38)
where \( B \) is a disk like mask with a small diameter: \( \text{diam } B \). Other operators that can be used to extract boundaries are

- Method I
  \[ (A \oplus B) - A \] (39)
- Method II
  \[ A - (A \odot B) \] (40)

Boundary detection using morphological operations is illustrated in figure 4.

### Region filling

Region filling can be accomplished by iteratively using dilation and intersection. We begin by defining a point \( p = 1 \) inside the hole, and we perform the following operation

\[ X_k = (X_{k-1} \oplus B) \cap A^c \] (41)
\[ X_0 = p \] (42)
where \( B \) is the cross structuring element. The algorithm exits when \( X_k = X_{k-1} \).
**Hit or miss transformation (hit and miss)**

Hit or miss is a method for template matching. It can be used to find the location of one shape among a set of shapes.

- Uses two structuring elements $B_1, B_2$, with $B_1 \cap B_2 = 0$.
- The hit or miss operation is
  \[
  A \odot B = (A \odot B_1) \cap (A^c \odot B_2)
  \]
  where $B = (B_1, B_2)$. Hit or miss is the intersection of two erosions, it uses a pair of positive and negative template. The result is 1 when $B_1$ fits in $A$ and $B_2$ misses $A$ (fits in $A^c$). The output is one $A$ has all pixels in $B_1$ but none in $B_2$.
- Hit or miss filter is widely used binary pattern recognition.

An illustration of using hit and miss to find a cross in $A$ is shown in figure 5.

**Majority filter**

A majority filter is a special case of the median filter. In a similar way to the median filter, the majority filter is very efficient in removing salt and pepper noise. An example is below:

- Original image:
  \[
  A = \begin{bmatrix}
  1 & 0 & 1 \\
  1 & 0 & 0 \\
  1 & 1 & 1 
  \end{bmatrix}
  \]

- The output image when the filter is applied to the center pixel with a 3 by 3 window is
  \[
  A = \begin{bmatrix}
  1 & 0 & 1 \\
  1 & 1 & 0 \\
  1 & 1 & 1 
  \end{bmatrix}
  \]