The Denavit-Hartenberg Convention

The Denavit-Hartenberg (DH) convention was introduced in 1955 by Jacques Denavit and Richard Hartenberg, it became popular only in the 1980’s. It is one of the most popular conventions used to study the kinematics of manipulators. It was introduced to reduce the complexity by selecting the reference frames. Clearly, it is used to represent the relationship between the individual joints and the end effector. Under the DH convention, each homogeneous transformation $A_i$ is represented by a product of four basic transformations as follows

$$A_i = Rot_z, θ_i Trans_z, d_i Trans_x, a_i Rot_x, α_i$$ (1)

where $θ_i, a_i, d_i, α_i$ are the DH parameters associated with link $i$ and joint $i$. They are called:
- $a_i$ is the link length
- $α_i$ is the link twist
- $d_i$ is the joint offset
- $θ_i$ is the joint angle

Recall that $A_i$ is a function of a single variable $q_i$ that we called the joint variable. This implies that amongst $θ_i, a_i, d_i, α_i$ there is only one variable and the others are constant parameters. The variable is $θ_i$ when we have a revolute joint and $d_i$ when we have a prismatic joint.

A. Reference frame

The way the reference frames are defined plays an important role in the DH convention. The axes are defined as follows:
- $z_i$ is the direction of the joint axis, i.e., the direction of rotation or translation.
- $x_i$ is along the common normal to $z_i−1$ and $z_i$ directed from $z_i−1$ to $z_i$. The common normal between two lines is the shortest line between them. Thus
  $$x_i = z_i \times z_i−1$$ (2)

$x_i$ is perpendicular to both $z_i$ and $z_i−1$ and intersects with both of them.
- $y_i$ is defined using the right hand rule (RHR).

B. Denavit-Hartenberg parameters

The DH-parameters are defined as follows:
- Joint offset $d_i$: distance from $z_{i−1}$ to $x_i$ measured along $z_i−1$. $d_i$ is the joint variable if joint $i$ is prismatic (joint $i$ in figure 1).
- Joint angle $θ_i$: angle between $x_{i−1}$ and $x_i$ about $z_{i−1}$. $θ_i$ is the joint variable if joint $i$ is revolute(joint $i−1$ in figure 1).
- Link length $a_i$: distance between $z_i$ and $z_{i−1}$ along $x_i$
- Link twist $α_i$: angle between $z_i$ and $z_{i−1}$ about $x_i$

Figure 1 illustrates the DH-convention and 2 shows the positive direction of rotation.

Example

Find the DH parameters for the planar manipulator of figure 3 and write $A_1$ and $A_2$ as a function of these parameters.

Solution

The DH-parameters are
Fig. 3. Example for DH parameters

<table>
<thead>
<tr>
<th>Link</th>
<th>$\theta$</th>
<th>$a$</th>
<th>$d$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\theta_1$</td>
<td>$a_1$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>$\theta_2$</td>
<td>$a_2$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

From which we get for $A_1$

$$A_1 = Rot_{z, \theta_1} Trans_{x, a_1} Trans_{x, a_2} Rot_{x, \alpha_1}$$  \hspace{1cm} (3)

with

$$Rot_{z, \theta_1} = \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 & 0 & 0 \\ \sin \theta_1 & \cos \theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$  \hspace{1cm} (4)

$$Trans_{z, d_1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$  \hspace{1cm} (5)

Since $d_1 = 0$, $Trans_{z, d_1}$ is just the identity matrix.

$$Trans_{x, a_1} = \begin{bmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$  \hspace{1cm} (6)

$$Rot_{x, \alpha_1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \alpha_1 & \sin \alpha_1 & 0 \\ -\sin \alpha_1 & \cos \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$  \hspace{1cm} (7)

Again $Rot_{x, \alpha_1} = I$ since $\alpha_1 = 0$. Now equation (3) becomes

$$A_1 = Rot_{z, \theta_1} Trans_{x, a_1} = \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 & 0 & 0 \\ \sin \theta_1 & \cos \theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$  \hspace{1cm} (8)

$$A_1 = \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 & 0 & a_1 \cos \theta_1 \\ \sin \theta_1 & \cos \theta_1 & 0 & a_1 \sin \theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$  \hspace{1cm} (9)

The same approach is used to find $A_2$. 

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