Robotics
Summary 11: Path planning methods

Path planning
Path planning is one of the most fundamental problems in robotics. Path planning has several applications in robotics and many other areas, such as video game design and the study of biological molecules. This summary discusses some of the most popular planning algorithms.

Terminology
- \( q \) is the configuration variable
- \( Q \) is the configuration space.

A. Example
For a revolute joint
- The configuration variable is \( q = \theta \), the angle
- The configuration space is \( Q = S^1 \), the circle, (set of angles \([0, 2\pi]\)).

- Complete algorithm: If there is a solution, the algorithm will find it, if there is no solution, the algorithm will let you know in finite time.

The potential field method
This is one of the most popular methods used for path planning. It was introduced in 1980s by Osama Khatib from Stanford University. In this method, the robot moves in an artificial potential field that is the sum of attractive and repulsive fields. The most important variables are \( U \) : the artificial potential field and \( F \) : the force resulting from \( U \). A more detailed description is below.

Principle of the method
- The goal location generates an attractive potential pulling the robot towards the goal
- The obstacles generate a repulsive potential pushing the robot away from the obstacles
- Artificial potential:
  \[
  U(q) = U_{\text{att}}(q) + U_{\text{rep}}(q)
  \]
- \( U_{\text{att}}(q) \): attractive potential.
- \( U_{\text{rep}}(q) \): repulsive potential.
- Artificial force:
  \[
  F(q) = -\nabla U(q)
  \]
  where
  \[
  \nabla = \begin{bmatrix}
  \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}
  \end{bmatrix}
  \]

Example: robot modeled as a point \((q = [x, y]^T)\)
  \[
  F(q) = -\nabla U(q) = \begin{bmatrix}
  \frac{-\partial U}{\partial x} \\
  \frac{-\partial U}{\partial y} \\
  \frac{-\partial U}{\partial z}
  \end{bmatrix}
  \]

Calculating the attractive force
One choice for the potential field is
  \[
  U_{\text{att},i}(q) = 0.5\zeta_i\|o_i(q) - o_i(q_f)\|^2
  \]
This field is referred to as the parabolic well potential. Note that \(o_i(q)\) is the origin of frame \(i\) attached to link \(i\), \(q_f\) is the desired final configuration. \(\zeta_i\) is a scaling parameter used to scale the effect of the attractive potential. The attractive force is given by
  \[
  F_{\text{att},i}(q) = -\nabla U_{\text{att},i}(q) = -\zeta_i(o_i(q) - o_i(q_f))
  \]
Equation 6 is a quadratic function of the distance. It is also possible to use a linear function of the distance. A quadratic function however presents several advantages, including simplicity of the calculations.

Calculating repulsive force
One choice for the potential field is
  \[
  U_{\text{rep},i}(q) = \begin{bmatrix}
  0.5\eta_i \left( \frac{1}{\rho(o_i(q))} - \frac{1}{\rho_0} \right)^2 & \frac{1}{\rho_0} \cdot 1 \cdot \frac{1}{\rho(o_i(q))} \\
  0 & 0
  \end{bmatrix}
  \]
where \(\rho_0\) is the distance of influence of the obstacle, \(\rho(o_i(q))\) shortest distance between \(o_i\) and the obstacle. \(\eta_i\) is a scaling parameter. The repulsive force for \(\rho(o_i(q)) \leq \rho_0\) is given by
  \[
  F_{\text{rep},i}(q) = \eta_i \left( \frac{1}{\rho(o_i(q))} - \frac{1}{\rho_0} \right) \frac{1}{\rho^2(o_i(q))} \nabla \rho(o_i(q))
  \]
where \(\nabla \rho(o_i(q))\) indicates the gradient calculated at \(o_i(q)\). For a point obstacle \(b\), \(\rho(o_i(q)) = \|o_i(q) - b\|\). Its gradient is
  \[
  \nabla \rho(o_i(q)) = \frac{o_i(q) - b}{\|o_i(q) - b\|}
  \]
The forces and the gradient are vectors. The potential is a scalar function.

B. Example
We want to plan the path for the two-link planar manipulator of figure 3 using the potential field method. The initial and final configurations are shown in the figure. Find the attractive force and represent (show) it in the figure.
C. Solution

From the figure, we have

\[
\begin{align*}
o_1(q_0) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
o_1(q_f) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
o_2(q_0) &= \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\
o_2(q_f) &= \begin{bmatrix} -1 \\ 1 \end{bmatrix}
\end{align*}
\]  

Using the equations for the attractive force, we get:

\[
\begin{align*}
F_{\text{att},1} &= -\zeta [o_1(q_0) - o_1(q_f)] = \zeta \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
F_{\text{att},2} &= -\zeta [o_2(q_0) - o_2(q_f)] = \zeta \begin{bmatrix} -3 \\ 1 \end{bmatrix}
\end{align*}
\]

The forces are represented in figure 3-top.

D. Example

Now, we consider the obstacle located at (2, 0.5) as shown in figure 3-bottom (we ignore the other obstacle for this example). We want to plan the path for the two link planar manipulator using the potential field method. The initial and final configurations are shown in the figure. Find the total force.

E. Solution

We take the threshold \( \rho_0 = 1 \). For this value, \( o_1 \) is not in the zone of influence of the obstacle, however, \( o_2 \) is. We calculate the repulsive force applied to \( o_1 \) and \( o_2 \).
1) For $o_1$, we have
   $$F_{rep,1} = 0$$

2) To calculate $F_{rep,2}$, we need to know $\rho(o_2(q))$ and $\nabla\rho(o_2(q))$. We have:
   $$o_2(q) - b = \begin{bmatrix} 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 0.5 \end{bmatrix}$$
   $$||o_2(q) - b|| = \sqrt{(2-2)^2 + (0-0.5)^2}$$

   Therefore,
   $$\nabla\rho(o_2(q)) = \begin{bmatrix} 0 \\ -0.5 \end{bmatrix} \frac{1}{\sqrt{(2-2)^2 + (0-0.5)^2}} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

   and
   $$F_{rep,2} = \eta_2(\frac{1}{0.5} - 1)(\frac{1}{0.25}) \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \eta_2 \begin{bmatrix} 0 \\ -4 \end{bmatrix}$$

For $\eta_2 = 1$ and $\zeta = 1$, the total force is
   $$F_{total,2} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -4 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$$

Figure 4 shows an illustration of the potential field method without obstacles. Figures 5 and 6 show an illustration of the potential field method in the presence of obstacles.

**PATH PLANNING USING PROPORTIONAL CONTROLLER**

Proportional controller is the simplest controller that can be used to track the desired configuration of a robotic system. The block diagram of a proportional controller is shown in figure 7, where $q_{des}$ is the desired configuration variable and $K$ is the gain. Clearly, this is a closed loop system. Assuming the sensor has no delay and that the actuator can be modeled using a first order system, the block diagram can be simplified as shown in figure 7-bottom, where $a$ is a constant that characterizes the transient response of the system (it also has an effect of the steady state error). We assume that $a$ is of the order of milliseconds. From the system block diagram, we can write:

\[
E = q_{des} - q = K \frac{e}{s + a} = \frac{K}{s + a}(q_{des} - q)
\]

and

\[
q = K \frac{e}{s + a} = \frac{K}{s + a}(q_{des} - q)
\]

\[
sq + aq = Kq_{des} - Kq
\]

\[
sq = -(K + a)q + Kq_{des}
\]

\[
q = -(K + a)q + Kq_{des}
\]

Now, it is possible to use the Euler approximation to approximate the time derivative. In general we have

\[
\dot{x} = \frac{x(k+1) - x(k)}{T}
\]
where \( k \) is the discrete time. Thus, for our particular case we have:

\[
\frac{q(k+1) - q(k)}{T} - (K + a)q + Kq_{des} = 0
\]  

(28)

And

\[
q(k+1) = (1 - T(a + K))q(k) + KTq_{des}
\]

(29)

F. Example

Figures 8 and 10 show the path of the planar manipulator to go from its initial configuration to two desired configurations \((\theta_1, \theta_2) = (45^\circ, -45^\circ)\) and \((\theta_1, \theta_2) = (45^\circ, 120^\circ)\), respectively. A proportional controller is used. Figures 9 and 11 show the time evolution of the configuration variables.