VELOCITY KINEMATICS AND JACOBIAN

The kinematics (and inverse kinematics) equations allowed us to write relationships between the position and orientation of the end effector and the position and orientation of the joint variables of the manipulator. The goal of the velocity kinematics is to establish a relationship between the linear and angular velocities of the end effector and those of the joint variables. Velocity kinematic is important for various reasons, including motion planning, achieving smooth motion and force and torque calculations.

JACOBIAN OF THE TWO LINK MANIPULATOR

For the two link planar manipulator of figure 1, the coordinates of the end effector are

\[ p_x = a_1 \cos \theta_1 + a_2 \cos(\theta_1 + \theta_2) \]
\[ p_y = a_1 \sin \theta_1 + a_2 \sin(\theta_1 + \theta_2) \]

By taking the time derivative, we get

\[ \dot{p}_x = -\dot{\theta}_1 a_1 \sin \theta_1 - \dot{\theta}_2 a_2 \sin(\theta_1 + \theta_2) \]
\[ \dot{p}_y = \dot{\theta}_1 a_1 \cos \theta_1 + \dot{\theta}_1 a_2 \cos(\theta_1 + \theta_2) + \dot{\theta}_2 a_2 \cos \theta_1 + \theta_2 \]

which can be re-arranged under matrix form:

\[
\begin{bmatrix}
\dot{p}_x \\
\dot{p}_y \\
\end{bmatrix} =
\begin{bmatrix}
-a_1 \sin \theta_1 - a_2 \sin(\theta_1 + \theta_2) & -a_2 \sin(\theta_1 + \theta_2) \\
-a_1 \cos \theta_1 + a_2 \cos(\theta_1 + \theta_2) & a_2 \cos \theta_1 + \theta_2 \\
\end{bmatrix}
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\end{bmatrix}
\]

In general, it is possible to write

\[
\begin{bmatrix}
v_x \\
v_y \\
\end{bmatrix} =
\begin{bmatrix}
\dot{p}_x \\
\dot{p}_y \\
\end{bmatrix} = J
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\end{bmatrix}
\]

with

\[
J =
\begin{bmatrix}
-a_1 \sin \theta_1 - a_2 \sin(\theta_1 + \theta_2) & -a_2 \sin(\theta_1 + \theta_2) \\
a_1 \cos \theta_1 + a_2 \cos(\theta_1 + \theta_2) & a_2 \cos \theta_1 + \theta_2 \\
\end{bmatrix}
\]

where \( J \) is the (partial) Jacobian matrix, it links the velocity of the end effector \( \dot{(\bar{p}_x, \bar{p}_y)} \) to the velocity of the joint variables \( \dot{(\theta_1, \theta_2)} \).

A. Example

At the configuration of the planar manipulator shown in figure 2 \( (\theta_1 = \theta_2 = 45^\circ) \), find the speed of the end effector when

\[
\dot{\theta}_1 = -32.41 \text{deg/s} \quad (8)
\]
\[
\dot{\theta}_2 = 75.38 \text{deg/s} \quad (9)
\]

knowing that \( a_1 = 5, a_2 = 4 \)

B. Solution

For the configuration of figure 2, the joint variables are

\[
\theta_1 = 45^\circ \quad (10)
\]
\[
\theta_2 = 45^\circ \quad (11)
\]

The Jacobian matrix calculated at this point is

\[
J =
\begin{bmatrix}
-7.5 & -4 \\
3.5 & 0 \\
\end{bmatrix}
\]
which gives the following speed for the end effector
\[
\begin{bmatrix}
v_x \\
v_y \\
v_z
\end{bmatrix} = \begin{bmatrix}
-1 \\
-2 \\
0
\end{bmatrix}
\]
(13)
The speed is represented in figure 2.

**DERIVATION OF THE JACOBIAN**

Consider an n-link manipulator with joint variables \(q_1, q_2, \ldots, q_n\), we define the velocity components as follows
- \(v^0_n\) is the linear velocity of the end effector
- \(\omega^0_n\) is the angular velocity of the end effector

It is possible to write
\[
v^0_n = J_v \dot{q}
\]
(14)
\[
\omega^0_n = J_\omega \dot{q}
\]
(15)
where \(J_v\) and \(J_\omega\) are \(3 \times n\) matrices. Combining the equations for the linear and angular velocities we have:
\[
\zeta = J \dot{q}
\]
with
\[
\zeta = \begin{bmatrix} v^0_n \\ \omega^0_n \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}
\]
(17)
and
\[
J = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix}
\]
(18)
Matrix \(J\) is the Jacobian matrix, its size \(6 \times n\).

**DERIVATION OF THE JACOBIAN**

C. **Angular velocity**
- Revolute joint: If we have a rotation about the z-axis, we have:
  \[
  \omega^i_{z_i-1} = \dot{q}^T \kappa
  \]
(19)
where \(\kappa = [0, 0, 1]^T\)
- Prismatic joint:
  \[
  \omega^i_{z_i-1} = 0
  \]
(20)

D. **Linear velocity**
- Revolute joint: We have:
  \[
  \omega = \dot{\theta} z_{i-1}
  \]
(21)
\[
\tau = a_i - a_{i-1}
\]
(22)
\[
J_{vi} = z_{i-1} \times (a_n - a_{i-1})
\]
(23)
- Prismatic joint: The direction of translation is \(z_{i-1}\)
  \[
  J_{vi} = z_{i-1}
  \]
(24)
A summary can be found below.

**Example**
Use the summary to find the two-link planar manipulator.

**Solution**

\[
J(q) = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix}
\]
(25)

with
\[
J(q) = \begin{bmatrix}
z_0 \times (a_2 - o_0) \\
z_1 \times (a_2 - o_1)
z_0
\end{bmatrix}
\]
(26)

where \(z_0, z_1\) represent the direction of rotation (about the z-axis in this case). From figure 1, the coordinates of the reference points are:
\[
o_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]
(27)
\[
o_1 = \begin{bmatrix} a_1 \cos \theta_1 \\ a_1 \sin \theta_1 \\ 0 \end{bmatrix}
\]
(28)
\[
o_2 = \begin{bmatrix} a_1 \cos \theta_1 + a_2 \cos(\theta_1 + \theta_2) \\ a_1 \sin \theta_1 + a_2 \sin(\theta_1 + \theta_2) \\ 0 \end{bmatrix}
\]
(29)

After calculations, we get
\[
J = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} = \begin{bmatrix}
-a_1 \sin \theta_1 - a_2 \sin(\theta_1 + \theta_2) & -a_2 \sin(\theta_1 + \theta_2) \\
a_1 \cos \theta_1 + a_2 \cos(\theta_1 + \theta_2) & a_2 \cos(\theta_1 + \theta_2)
\end{bmatrix}
\]
(30)
\[
J^{-1} \zeta = J^{-1} J \dot{q}
\]
(31)
\[
\dot{q} = J^{-1} \zeta
\]
(32)

This confirms the previous results.

**INVERSE VELOCITY KINEMATICS**

The inverse Jacobian allows to calculate the linear and angular velocity of the joint variables knowing the velocity of the end effector. Three different cases are discussed below
- Case 1: The Jacobian matrix is invertible

A solution exists only if \(J\) is invertible. One condition is that \(J\) must be a square matrix, which implies that \(n = 6\).
• Case 2: \( n > 6 \): There are fewer constraints than unknowns. Several choices of \( \dot{q} \) can lead to the same \( \zeta \). The solution has \( n - 6 \) degrees of freedom. The solution can be obtained using the pseudoinverse matrix as follows

\[
\zeta = J \dot{q} \tag{35}
\]

\[
J^T \zeta = J^T J \dot{q} \tag{36}
\]

\[
\dot{q} = (J^T J)^{-1} J^T \zeta \tag{37}
\]

• Case 3: \( n < 6 \): More equations than unknowns. There is no solution in general.

\section*{E. Example}

Find the velocity of the joints that allows to move the end effector

- Case 1: Horizontally with speed 1\( \text{ms} \) at configuration \((\theta_1, \theta_2) = (45^\circ, 45^\circ)\)
- Case 2: Vertically with speed 2\( \text{ms} \) at configuration \((\theta_1, \theta_2) = (45^\circ, 90^\circ)\)

\section*{F. Solution}

- Case 1:
  The inverse Jacobian is calculated first

\[
J^{-1} = \begin{bmatrix}
0 & 0.28 \\
-0.25 & -0.53
\end{bmatrix} \tag{38}
\]

from which the velocity of the joints is obtained

\[
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2
\end{bmatrix} = \begin{bmatrix}
0^\circ \\
-14.3^\circ
\end{bmatrix} \tag{39}
\]

- Case 2:
  The inverse Jacobian is

\[
J^{-1} = \begin{bmatrix}
-0.14 & 0.14 \\
-0.03 & -0.32
\end{bmatrix} \tag{40}
\]

from which the velocity of the joints is obtained

\[
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2
\end{bmatrix} = \begin{bmatrix}
16.2^\circ \\
-36.4^\circ
\end{bmatrix} \tag{41}
\]

Cases 1 and 2 are illustrated in figures 3 and 4, respectively.
Summary:
The upper half of the Jacobian, $J_v$, is given by

\[ J_v = [J_{v1}, ..., J_{vn}] \]  \hspace{1cm} (42)

in which the $i^{th}$ column $J_{vi}$ is given by

\[ J_{vi} = \begin{cases} 
    z_i - 1 \times (o_n - o_{i-1}) & \text{for revolute joint } i \\
    z_{i-1} & \text{for prismatic joint } i
\end{cases} \] \hspace{1cm} (43)

The lower half of the Jacobian, $J_\omega$, is given by

\[ J_\omega = [J_{\omega1}, ..., J_{\omega n}] \] \hspace{1cm} (44)

in which the $i^{th}$ column $J_{\omega i}$ is given by

\[ J_{\omega i} = \begin{cases} 
    z_{i-1} & \text{for revolute joint } i \\
    0 & \text{for prismatic joint } i
\end{cases} \] \hspace{1cm} (45)

NOTE
For prismatic joint, $z_{i-1}$ is the direction of translation.