I. PROPORTIONAL CONTROLLER FOR SECOND ORDER SYSTEMS

The transfer function of a proportion controller is a pure gain. We first begin with a continuous time example and we generalize our approach to discrete time systems in the z-domain. The transfer function of proportional control is

\[ C(s) = K \]  

An example of using proportional controller to control the closed loop system is shown below.

A. Example:

A position control system has the following transfer function

\[ G(s) = \frac{Ks}{s(s+4)} \]  

Design a proportional controller for the system to obtain

1) a specified damping ratio, \( \zeta = 0.77 \)
2) a specified undamped natural frequency, \( \omega_n = 3.25 \)

The first step is to obtain the closed loop transfer function, which is given by:

\[ G_{cl}(s) = \frac{K}{s^2 + 4s + K} \]  

Now, we match this transfer function with the standard form characteristic polynomial:

\[ s^2 + 4s + K = s^2 + 2\zeta\omega_n s + \omega_n^2 \]  

1) For damping ratio of \( \zeta = 0.77 \), we have

\[ 2\zeta\omega_n = 4, \text{ and } \omega_n^2 = K \]  

and the solution for the gain is given by:

\[ K = \frac{4}{\zeta^2} = 6.75 \]  

Note that in this case, we cannot choose \( \omega_n \), it is constrained by its relationship with \( K \).

2) For an undamped natural frequency of \( \omega_n = 3.25 \), we have

\[ K = \omega_n^2 = 10.56 \]  

The corresponding value for the damping ratio is \( \zeta = 0.6154 \).

The step response for different values of the gain is shown in figure 1. Clearly, achieving desired values for both the damping ratio and the undamped natural frequency using a proportional controller is constrained. The code allowing to obtain and plot the closed loop response is as follows:

\[ d0 = tf([1, 1.4 0]) \]  

\[ CL2 = feedback(2 * d0, 1) \]  

\[ CL10 = feedback(10 * d0, 1) \]  

\[ CL15 = feedback(15 * d0, 1) \]  

\[ step(CL2), \text{ hold on} \]  

\[ step(CL10), \text{ hold on} \]  

\[ step(CL15), \text{ hold on} \]

Matlab command feedback does the closed loop transfer function.

Desired characteristics of a control system include stability, minimizing steady state error and desired transient. Last time we saw that the standard form for continuous time second order systems is given by

\[ G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]  

The behavior of the closed loop system depends on the location of the poles. Our goal is to develop similar equations by mapping the s-domain to the z-domain. The continuous domain poles are

\[ s_{1,2} = -\zeta\omega_n \pm j\omega_n \sqrt{1 - \zeta^2} \]  

We already know that

\[ z = e^{sT} \]
where $T$ is the sampling time. We can write for the discrete time poles:

$$z_{1,2} = e^{s_{1,2}}$$

$$= e^{-\zeta \omega_n T} e^{\pm j\omega_d T}$$

$$= e^{-\zeta \omega_n T} e^{\pm j\omega_d T}$$

which can be written as

$$z_{1,2} = |z| \angle \pm \theta$$

with

$$|z| = e^{-\zeta \omega_n T}$$

and

$$\theta = \omega_d T = \omega_n T \sqrt{1 - \zeta^2}$$

Now, our goal is to find the discrete time characteristic polynomial. Knowing that the poles are $z_1$ and $z_2$, the characteristic polynomial is given by

$$Q(z) = (z - z_1)(z - z_2)$$

$$= (z - e^{-\zeta \omega_n T} e^{j\omega_d T})(z - e^{-\zeta \omega_n T} e^{-j\omega_d T})$$

$$= z^2 - 2 \cos(\omega_d T) e^{-\zeta \omega_n T} z + e^{-2\zeta \omega_n T}$$

II. PROPORTIONAL CONTROLLER IN THE Z-DOMAIN

Now we consider designing a proportional controller to achieve the following goals: (1) desired transient, and (2) desired steady state error.

A. Example 1

Consider the following open loop transfer function in the $z$-domain

$$G(z) = \frac{1}{(z - 1)(z - 0.3)}$$

With $T = 0.1s$. Design a proportional controller to achieve the following

1) A damped natural frequency of 5 rad/s
2) A settling time of 2s.

First, we find the closed loop characteristic equation. The closed loop transfer function is given by

$$G_{cl}(z) = Z - \frac{K}{z^2 - 1.3z + 0.3 + K}$$

In order to solve, we need to match the coefficients in the characteristic polynomial with the standard form characteristic polynomial. This will give us:

$$1.3 = 2 e^{-\zeta \omega_n T} \cos(\omega_d T)$$

$$K + 0.3 = e^{-2\zeta \omega_n T}$$

1) For $\omega_d = 5 \text{rad/s}$, we have

$$e^{-2\zeta \omega_n T} = \left(\frac{1.3}{2 \cos(\omega_d T)}\right)^2 = K + 0.3$$

Solving for the gain yields $K = 0.248$

2) Settling time of 2s.

$$\frac{4}{\omega_n \zeta} = 2$$

$$\omega_n \zeta = 2 \text{ rad/s}$$

Using the second equation, we get

$$K = e^{-2(2)0.1} - 0.3 = 0.37$$

Figure 2 shows the closed loop step response for this value of the gain, which confirms the settling time. Solving for the damping ratio analytically is difficult. However, it is easy to solve numerically or graphically using the root locus. In order to solve, we need to plot the contours of constant values of $\zeta$ in the root locus. This can be done using function $\text{zgrid}$ as follows

$$\text{Rlocus}(L), \quad \text{zgrid},$$

where $L$ is the open loop transfer function. This command gives curves corresponding to constant $\zeta$ and curves corresponding to constant $\omega_n$. The next step is to click on the intersection of the root locus with the desired $\zeta$ or $\omega_n$. An illustration is shown in figure 3, where a damping ratio of $\zeta = 0.3$ corresponds to a gain $K = 0.366$. This point also corresponds to a natural frequency of 6.91 rad/s. The next example deals with designing a proportional controller to control the steady state.

B. Example 2

For the previous system where

$$G(z) = \frac{0.47}{z - 0.43}$$
design a proportional controller so that the steady state error to a ramp is less than 20%. Clearly, the system is of type 1.

\[ K_v = \lim_{z \to 1} \frac{1}{T} K(z - 1) L(z) \]  
\[ = \frac{1}{T} \frac{K}{z - 0.3} \]  
\[ = \frac{K}{0.1 \cdot 0.7} \]  
\[ = \frac{K}{0.07} \]  
\[ e(\infty) = \frac{1}{K_v} = 0.07 = 0.2 \]  
\[ K > 0.35. \]

### Reducing System Order

Sometimes higher order systems can be approximated by first or second order systems. Consider for example

\[ G(z) = \frac{z}{(z - 0.1)(z - 0.2)(z - 0.9)} \]  
\[ G_1(z) = \frac{z}{z - 0.9} \]  
\[ G_2(z) = \frac{z}{(z - 0.2)(z - 0.9)} \]

The poles have different effects on the system’s response. By taking the inverse z-transform, it is possible to write

\[ g(k) = a \cdot 0.1^k + b \cdot 0.2^k + c \cdot 0.9^k \]

where \( a, b, c \) are constants. when \( k = 3 \), it is possible to write

\[ g(k) = a \cdot 0.001 + b \cdot 0.008 + c \cdot 0.729 \]

After three iterations only, it can be seen that the effect of the first and second pole is reduced considerably compared to the effect of the third pole. The third pole is called the dominant pole. Therefore, it is possible to reduce the order of the system by keeping the dominant pole only, which is the slowest pole in the system. An example is shown below.

### Example

Consider the third order transfer function given by

\[ G(z) = \frac{z}{(z - 0.1)(z - 0.2)(z - 0.9)} \]  

- By keeping the dominant pole only, we get the first order system given by

\[ G_1(z) = \frac{z}{z - 0.9} \]

- By keeping two dominant poles, we get the second order system given by

\[ G_2(z) = \frac{z}{(z - 0.2)(z - 0.9)} \]

The step response of the three systems is shown in figure 4.

The poles have an effect on the system’s transient response but also on its final value as shown in figure 4. The reduced order systems do not have the same final value as the original system. In order to fix the problem we perform gain adjustment so that the reduced order systems have the same final value as the original system. Let \( K \) and \( K_1 \) be the DC gains (\( z = 1 \)) of the original system and the reduced order system, respectively, and \( G_{apr} \) is the reduced order transfer function. In order to have the same final value, the approximation must be scaled as follows

\[ G_{apr} \leftarrow \frac{K}{K_i} G_{apr} \]  

This simple gain adjustment allows for \( G_{apr} \) and the original system to have the same final value.

### Example

For the previous example, the DC gains are

\[
\begin{bmatrix}
\text{System} & \text{DC Gain} \\
G(z) & K = 13.89 \\
G_1(z) & K_1 = 10 \\
G_2(z) & K_2 = 12.5 \\
\end{bmatrix}
\]

Therefore,

\[ G_1(z) = 1.389 \frac{z}{z - 0.9} \]  
\[ G_2(z) = 1.1112 \frac{z}{(z - 0.2)(z - 0.9)} \]

The systems’ response after gain adjustment is shown in figure 5.

### NOTE

The s-plane can be represented as \( s = \sigma + j\omega \). The relationship between the s and z-planes is

\[ z = e^{\sigma T} e^{j\omega T} = |z| e^{j\omega T} \]
Constant values of $\sigma$ correspond to circles in the $z$-plane with radius $|z|$.

- The origin of the $s$-plane ($s = 0$) is mapped to $z = e^0 = 1$ on the real axis in the $z$-plane.
- The left most vertical line $\sigma = -\infty$ is mapped to the origin of the $z$-plane $|z| = e^{-\infty} = 0$.
- The right most vertical line $\sigma = \infty$ is mapped to circle of infinite radius $|z| = e^{\infty} = \infty$.
- The imaginary axis $\sigma = 0$ is mapped to a circle with radius 1, i.e., the unit circle, $|z| = e^0 = 1$. 

Fig. 4. Third order system and approximation with lower order systems without gain adjustment

Fig. 5. Third order system and approximation with lower order systems with gain adjustment